Let $R$ be a ring with 1 having radical (Jacobson) $N$. $R$ is called semiprimary \cite[p. 56]{2} if and only if $R/N$ satisfies the minimum condition for right ideals. If $M$ is a right $R$-module, a submodule $A$ of $M$ is called small \cite{5} if $A + B = M$ for any submodule $B$ of $M$ implies $B = M$. A submodule $A$ of $M$ is called large \cite{3} if $A \cap B = 0$ for any submodule $B$ of $M$ implies $B = 0$. A right ideal in $R$ is called small or large if it is small or large as a submodule of the right regular $R$-module $R_R$. A projective cover \cite{1} of $M$ is an epimorphism of a projective module onto $M$ such that its kernel is small. The main results of this paper are the following theorems:

**Theorem 1.** Every irreducible (right) $R$-module has a projective cover if and only if $R$ is semiprimary and for any nonzero idempotent $x + N$ in $R/N$ there is a nonzero idempotent $e$ in $R$ such that $ex - e \in N$.

Theorem 1 is related to Theorem 2.1 of \cite{1}.

**Theorem 2.** If $R$ is commutative then every irreducible $R$-module has a projective cover if and only if $R$ is semiprimary and for any nonzero idempotent $x + N$ in $R/N$ there is an idempotent $e \in R$ such that $x - e \in N$.

**Lemma 1.** If $I$ is a maximal right ideal of $R$ then the right $R$-module $R/I$ has a projective cover if and only if there is a nonzero idempotent $e \in R$ such that $eI$ is small.

**Proof.** Let $f$ be an epimorphism from a projective module $P$ onto $R/I$ such that the kernel of $f$ is small in $P$. Since $R$ is projective (as $R_R$), there is an $R$-homomorphism $h$ from $R$ into $P$ making

\[
\begin{array}{ccc}
R & \xrightarrow{h} & P \\
\downarrow{\pi} & & \downarrow{f} \\
R/I & \xrightarrow{0} & 0
\end{array}
\]

where $\pi$ is the natural mapping, commutative. Since for any arbitrary $p \in P$, $f(p) = \pi(x) = fh(x)$ for some $x \in R$, $p - h(x) \in \text{Ker } f$. Hence

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$P = \text{Ker } f + h(R)$. Since the Ker $f$ is small, this implies that $P = h(R)$.
Let $p_0 = h(1)$. Then $P = p_0R$ and $R^{p_0}p_0R \rightarrow 0$, where $t_{p_0}(x) = p_0x$ for all $x \in R$, is direct since $P$ is projective. Hence $\text{Ker } t_{p_0} = \{ r \in R \mid p_0r = 0 \}$ is a direct summand of $R$. Since $p_0 = h(1)$, Ker $h = \text{Ker } t_{p_0}$. If $h(I) = 0$, then Ker $t_{p_0} = I$ and $I$ is a direct summand of $R$. Hence there is a minimal right ideal $J$ in $R$ such that $R = J \oplus I$. Thus, by [2, p. 50], there is an idempotent $e \neq 0$ in $J$ such that $eI = 0$ is small. If $h(I) \neq 0$ then $h(I) \subset \text{Ker } f$ since $fh(I) = \pi(I) = 0$. Thus $h(I)$ is small. Since $h(R)$ is projective, there is an $R$-homomorphism $\phi$ from $h(R)$ making

\[
\phi
\]

where $i$ is the identity map, commutative. Since $h(I)$ is small, $\phi(h(I))$ is small in $R$ by [4, p. 93]. Let $\phi(p_0) = a \in R$. Then $p_0 = h\phi(p_0) = h(a) = h(1)a = p_0a$. Therefore, $a = \phi(p_0) = \phi(p_0a) = a^2$ and $aI = \phi(h(I))$ is small. Clearly $a \neq 0$ since $h\phi(p_0) = p_0$. Conversely, suppose there is a nonzero idempotent $e$ in $R$ such that $eI$ is small. Since $eI \subseteq N$ by [1, Lemma 2.4], the right ideal $(I : e) = \{ r \in R \mid erI \}$ is $I$. Define a mapping $g$ from $eR$ onto $R/I$ by $g(er) = r + I$ for all $er \in eR$. Since $er_1 = er_2$ then $r_1 - r_2 \in (I : e) = I$, $g$ is well defined and clearly $g$ is an $R$-homomorphism from $eR$ onto $R/I$. Furthermore since $eR$ is a direct summand of $R$, $eR$ is projective and since the kernel of $g$ is $eI$, which is small, $g$ is a projective cover for $R/I$.

**Lemma 2.** Let $I$ be a large maximal right ideal in $R$ and let $L = \{ x \in R \mid xI = 0 \}$. Then $L^2 = 0$.

**Proof.** If $x \neq 0, y \neq 0$ are elements in $L$ then $I \cap yR \neq 0$ and $x(yr) = 0$ for some $r \in R$ such that $yr \neq 0$ in $I$. If $xy \neq 0$, then $r \in I$ since the set $\{ r \in R \mid (xy)r = 0 \} = I$. This is impossible since $yr \neq 0$ and $y \in L$. Thus $L^2 = 0$.

**Proof of Theorem 1.** Suppose every irreducible $R$-module has a projective cover. Let $\overline{I}$ be a maximal right ideal of $R/N$. Then there is a maximal right ideal $I$ in $R$ such that $\overline{I} = I/N$. By Lemma 1, there is a nonzero idempotent $e$ in $R$ such that $eI$ is small. By [1, Lemma 2.4], $eI \subseteq N$. Since $e \subseteq N$, $e + N$ is a nonzero left annihilator of $\overline{I}$. Hence by Lemma 2, $\overline{I}$ cannot be large. Since $\overline{I}$ is a maximal right ideal of $R/N$, $\overline{I}$ must be a direct summand of $R/N$ if $\overline{I}$ is not large. Thus by
[6, Lemma 3.1], $R/N$ must be a semisimple ring with the minimum condition for right ideals. Now let $x \in R$ such that $x^2 - x \in N$. If $x \in N$, by Zorn's Lemma, we can construct a right ideal $J^*$ in $R$ with the properties that $N \subseteq J^*$, $x \in J^*$ such that if $K$ is a right ideal which contains $J^*$ properly then $x \in K$. Then the right $R$-module $xR + J^*/J^*$ is irreducible and $\langle J^*: x \rangle = \{ r \in R | xr \in J^* \}$ is a maximal right ideal of $R$. Hence there is an idempotent $e \neq 0$ in $R$ such that $e \cdot (J^*: x) \subseteq N$. Since $x^2 - x = x(x - 1) \in N$, $(x - 1) \in (J^*: x)$ and $e(x - 1) = ex - e \in N$.

Conversely, suppose $R$ is semiprimary and if $x + N$ is a nonzero idempotent in $R/N$ then there is a nonzero idempotent $e$ in $R$ such that $ex - e \in N$. If $I$ is a maximal right ideal of $R$, $I/N$ is a maximal right ideal of $R/N$, and since $R$ is semiprimary, there is a minimal right ideal $K/N$ in $R/N$ such that $K/N \cap I/N = N$ and $K/N \oplus I/N = R/N$ (see [4, p. 67]). Let $x = x + N$, for some $x \in R$, be a nonzero idempotent in $K/N$ such that $x \cdot (I/N) = N$. By hypothesis, there is a nonzero idempotent $e$ in $R$ such that $ex - e \in N$. Since $xI \subseteq N$ and $ex - e \in N$, $eI \subseteq N$. Thus by Lemma 1, $R/I$ has a projective cover.

The following corollary is related to Corollary 1 of [4, p. 76].

**Corollary.** A ring $R$ is local (i.e. $R/N$ is a division ring) if and only if 1 is a primitive idempotent and every irreducible $R$-module has a projective cover.

**Proof.** If $R$ is a local ring then 1 and 0 are only idempotents in $R$, and since $N$ is the only maximal right (left) ideal in $R$, every irreducible $R$-module has a projective cover. Conversely, suppose every irreducible $R$-module has a projective cover and 1 is a primitive idempotent in $R$. By Theorem 1, $R/N$ is a semisimple ring with the minimum condition on right ideals and if $x + N$ is a nonzero idempotent in $R/N$ there is a nonzero idempotent $e$ in $R$ such that $ex - e \in N$. Since 1 is a primitive idempotent in $R$, $e = 1$. Hence only idempotents in $R/N$ are zero and $1 + N$. Since $R/N$ is semisimple with a minimal right ideal, this implies that $R$ is a local ring.

**Proof of Theorem 2.** We only need to prove that if $R$ is commutative such that every irreducible $R$-module has a projective cover then idempotents modulo $N$ can be lifted. We first prove that if $x + N$ is an idempotent such that $(x + N)(R/N)$ is a minimal ideal in $R/N$ then $x - e \in N$ for some idempotent $e$ in $R$. Let $J^*$ be as in the proof of Theorem 1. Since $xR + N \supseteq J^* \supseteq N$ and $xR + N/N$ is a minimal ideal of $R/N$, $J^* = N$ since $J^*$ is properly contained in $xR + N$. As in the case of the proof of Theorem 1, there is an idempotent $e$ in $R$ such that $e \cdot (J^*: x) = e \cdot (N: x) \subseteq N$. Now $(N: ex) = (N: x) = (N: e)$ since $(N: x)$ is a maximal ideal and $(N: ex) \supseteq (N: x) \supseteq (N: e) \supseteq (N: ex)$. Thus
\[(1 - e) \in (N: e) = (N: x) \text{ and } x - xe \in N. \text{ Since } ex - e \in N, \text{ this implies that } x - e \in N. \text{ Now let } g = g^2 \text{ in } R \text{ such that } xg \in N. \text{ Since } e - x \in N, eg \in N. \text{ Let } e' = e - eg. \text{ Then } g \cdot e' = 0 \text{ and } e' \cdot e' = (e - eg)(e - eg) = e - eg - eg + eg = e', e' + N = e + N = x + N. \text{ It is well known that if } R/N \text{ is a semisimple ring with the minimum condition then } 1 + N = (x_1 + N) + (x_2 + N) + \cdots + (x_n + N) \text{ for some positive integer } n \text{ where } x_i - x_i^2 \in N, i = 1, 2, \ldots, n, x_i x_j \in N \text{ if } i \neq j \text{ and } (N: x_i), \text{ for each } i, \text{ is a maximal right ideal (see [2, p. 46 and p. 50]). By the above argument, we can choose an orthogonal set of idempotents } e_1, e_2, \ldots, e_n \text{ in } R \text{ such that } x_i - e_i \in N, i = 1, 2, \ldots, n, \text{ and } 1 + N = (e_1 + N) + (e_2 + N) + \cdots + (e_n + N). \text{ Now let } y + N \text{ be an arbitrary nonzero idempotent in } R/N. \text{ Then } y + N = (e_1 y + N) + (e_2 y + N) + \cdots + (e_n y + N) \text{ and } e_i y \cdot e_j y \in N \text{ if } i \neq j \text{ and } (N: e_i y) \text{ is a maximal ideal for all } i \text{ such that } e_i y \in N. \text{ There is an orthogonal set of idempotents } a_1, a_2, \ldots, a_n \text{ in } R \text{ such that } y - (a_1 + a_2 + \cdots + a_n)^2 \in N \text{ and } (a_1 + a_2 + \cdots + a_n)^2 = (a_1 + a_2 + \cdots + a_n). \]

References


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