ON A SEMIPRIMARY RING

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Let $R$ be a ring with 1 having radical (Jacobson) $N$. $R$ is called semiprimary [2, p. 56] if and only if $R/N$ satisfies the minimum condition for right ideals. If $M$ is a right $R$-module, a submodule $A$ of $M$ is called small [5] if $A + B = M$ for any submodule $B$ of $M$ implies $B = M$. A submodule $A$ of $M$ is called large [3] if $A \cap B = 0$ for any submodule $B$ of $M$ implies $B = 0$. A right ideal in $R$ is called small or large if $I$ is small or large as a submodule of the right regular $R$-module $R_R$. A projective cover [1] of $M$ is an epimorphism of a projective module onto $M$ such that its kernel is small. The main results of this paper are the following theorems:

**Theorem 1.** Every irreducible (right) $R$-module has a projective cover if and only if $R$ is semiprimary and for any nonzero idempotent $x+N$ in $R/N$ there is a nonzero idempotent $e$ in $R$ such that $ex - e \in N$.

Theorem 1 is related to Theorem 2.1 of [1].

**Theorem 2.** If $R$ is commutative then every irreducible $R$-module has a projective cover if and only if $R$ is semiprimary and for any nonzero idempotent $x+N$ in $R/N$ there is an idempotent $e \in R$ such that $x - e \in N$.

**Lemma 1.** If $I$ is a maximal right ideal of $R$ then the right $R$-module $R/I$ has a projective cover if and only if there is a nonzero idempotent $e \in R$ such that $eI$ is small.

**Proof.** Let $f$ be an epimorphism from a projective module $P$ onto $R/I$ such that the kernel of $f$ is small in $P$. Since $R$ is projective (as $R_R$), there is an $R$-homomorphism $h$ from $R$ into $P$ making

\[
\begin{array}{ccc}
R & \xrightarrow{h} & P \\
\downarrow \pi & & \downarrow f \\
R/I & \xrightarrow{\pi} & 0 \\
\end{array}
\]

where $\pi$ is the natural mapping, commutative. Since for any arbitrary $p \in P$, $f(p) = \pi(x) = fh(x)$ for some $x \in R$, $p - h(x) \in \text{Ker } f$. Hence

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\( P = \text{Ker} f + h(R). \) Since the \( \text{Ker} f \) is small, this implies that \( P = h(R). \)

Let \( p_0 = h(1). \) Then \( P = p_0R \) and \( R^{f_{p_0}} p_0R \rightarrow 0, \) where \( t_{p_0}(x) = p_0x \) for all \( x \in R, \) is direct since \( P \) is projective. Hence \( \text{Ker} t_{p_0} = \{ r \in R \mid p_0r = 0 \} \) is a direct summand of \( R. \) Since \( p_0 = h(1), \) \( \text{Ker} h = \text{Ker} t_{p_0}. \)

If \( h(I) = 0, \) then \( \text{Ker} t_{p_0} = I \) and \( I \) is a direct summand of \( R. \) Hence there is a minimal right ideal \( J \) in \( R \) such that \( R = J \oplus I. \) Thus, by \([2, \text{p. 50}],\) there is an idempotent \( e \neq 0 \) in \( J \) such that \( eI = 0 \) is small. If \( h(I) \neq 0 \) then \( h(I) \subset \text{Ker} f \) since \( fh(I) = \pi(I) = 0. \) Thus \( h(I) \) is small. Since \( h(R) \) is projective, there is an \( R \)-homomorphism \( \phi \) from \( h(R) \) making

\[
\begin{array}{ccc}
R & \xrightarrow{i} & h(R) \\
\phi \downarrow & & \downarrow h \\
\end{array}
\]

where \( i \) is the identity map, commutative. Since \( h(I) \) is small, \( \phi(h(I)) \) is small in \( R \) by \([4, \text{p. 93}],\) Let \( \phi(p_0) = a \in R. \) Then \( p_0 = h\phi(p_0) = h(a) = h(1)a = p_0a. \) Therefore, \( a = \phi(p_0) = h(p_0a) = a^2 \) and \( aI = \phi(h(I)) \) is small. Clearly \( a \neq 0 \) since \( h\phi(p_0) = p_0. \) Conversely, suppose there is a nonzero idempotent \( e \) in \( R \) such that \( eI \) is small. Since \( eI \subseteq N \) by \([1, \text{Lemma 2.4}],\) the right ideal \( (I: e) = \{ r \in R \mid er \in I \} \) is \( I. \) Define a mapping \( g \) from \( eR \) onto \( R/I \) by \( g(er) = r + I \) for all \( er \in eR. \) Since \( er_1 = er_2 \) then \( r_1 - r_2 \in (I: e) = I, \) \( g \) is well defined and clearly \( g \) is an \( R \)-homomorphism from \( eR \) onto \( R/I. \) Furthermore since \( eR \) is a direct summand of \( R, \) \( eR \) is projective and since the kernel of \( g \) is \( eI, \) which is small, \( g \) is a projective cover for \( R/I. \)

**Lemma 2.** Let \( I \) be a large maximal right ideal in \( R \) and let \( L = \{ x \in R \mid xI = 0 \}. \) Then \( L^2 = 0. \)

**Proof.** If \( x \neq 0, y \neq 0 \) are elements in \( L \) then \( I \cap yR \neq 0 \) and \( x(yr) = 0 \) for some \( r \in R \) such that \( yr \neq 0 \) in \( I. \) If \( xy \neq 0, \) then \( r \in I \) since the set \( \{ r \in R \mid (xy)r = 0 \} = I. \) This is impossible since \( yr \neq 0 \) and \( y \in L. \) Thus \( L^2 = 0. \)

**Proof of Theorem 1.** Suppose every irreducible \( R \)-module has a projective cover. Let \( \overline{I} \) be a maximal right ideal of \( R/N. \) Then there is a maximal right ideal \( I \) in \( R \) such that \( \overline{I} = I/N. \) By Lemma 1, there is a nonzero idempotent \( e \) in \( R \) such that \( eI \) is small. By \([1, \text{Lemma 2.4}],\) \( eI \subseteq N. \) Since \( e \subseteq N, \) \( e + N \) is a nonzero left annihilator of \( \overline{I}. \) Hence by Lemma 2, \( \overline{I} \) cannot be large. Since \( \overline{I} \) is a maximal right ideal of \( R/N, \) \( \overline{I} \) must be a direct summand of \( R/N \) if \( \overline{I} \) is not large. Thus by
[6, Lemma 3.1], \( R/N \) must be a semisimple ring with the minimum condition for right ideals. Now let \( x \in R \) such that \( x^2 - x \in N \). If \( x \in N \), by Zorn's Lemma, we can construct a right ideal \( J^* \) in \( R \) with the properties that \( N \subseteq J^* \), \( x \in J^* \) such that if \( K \) is a right ideal which contains \( J^* \) properly then \( x \in K \). Then the right \( R \)-module \( xR + J^*/J^* \) is irreducible and \( (J^*: x) = \{ r \in R \mid xr \in J^* \} \) is a maximal right ideal of \( R \). Hence there is an idempotent \( e \neq 0 \) in \( R \) such that \( e \cdot (J^*: x) \subset N \). Since \( x^2 - x = x(x - 1) \in N \), \( (x - 1) \in (J^*: x) \) and \( e(x - 1) = ex - e \in N \). Conversely, suppose \( R \) is semiprimary and if \( x + N \) is a nonzero idempotent in \( R/N \) then there is a nonzero idempotent \( e \) in \( R \) such that \( ex - e \in N \). If \( I \) is a maximal right ideal of \( R \), \( I/N \) is a maximal right ideal of \( R/N \), and since \( R \) is semiprimary, there is a minimal right ideal \( K/N \) in \( R/N \) such that \( K/N \cap I/N = N \) and \( K/N \oplus I/N = R/N \) (see [4, p. 67]). Let \( \tilde{x} = x + N \), for some \( x \in R \), be a nonzero idempotent in \( K/N \) such that \( \tilde{x} \cdot (I/N) = N \). By hypothesis, there is a nonzero idempotent \( e \) in \( R \) such that \( ex - e \in N \). Since \( xI \subset N \) and \( ex - e \in N \), \( eI \subset N \). Thus by Lemma 1, \( R/I \) has a projective cover.

The following corollary is related to Corollary 1 of [4, p. 76].

**Corollary.** A ring \( R \) is local (i.e. \( R/N \) is a division ring) if and only if 1 is a primitive idempotent and every irreducible \( R \)-module has a projective cover.

**Proof.** If \( R \) is a local ring then 1 and 0 are only idempotents in \( R \), and since \( N \) is the only maximal right (left) ideal in \( R \), every irreducible \( R \)-module has a projective cover. Conversely, suppose every irreducible \( R \)-module has a projective cover and 1 is a primitive idempotent in \( R \). By Theorem 1, \( R/N \) is a semisimple ring with the minimum condition on right ideals and if \( x + N \) is a nonzero idempotent in \( R/N \) there is a nonzero idempotent \( e \) in \( R \) such that \( ex - e \in N \). Since 1 is a primitive idempotent in \( R \), \( e = 1 \). Hence only idempotents in \( R/N \) are zero and \( 1 + N \). Since \( R/N \) is semisimple with a minimal right ideal, this implies that \( R \) is a local ring.

**Proof of Theorem 2.** We only need to prove that if \( R \) is commutative such that every irreducible \( R \)-module has a projective cover then idempotents modulo \( N \) can be lifted. We first prove that if \( x \in R \) is an idempotent such that \( (x + N)(R/N) \) is a minimal ideal in \( R/N \) then \( x - e \in N \) for some idempotent \( e \) in \( R \). Let \( J^* \) be as in the proof of Theorem 1. Since \( xR + N \supset J^* \supset N \) and \( xR + N/N \) is a minimal ideal of \( R/N \), \( J^* = N \) since \( J^* \) is properly contained in \( xR + N \). As in the case of the proof of Theorem 1, there is an idempotent \( e \) in \( R \) such that \( e \cdot (J^*: x) = e \cdot (N: x) \subset N \). Now \( (N: ex) = (N: x) = (N: e) \) since \( (N: x) \) is a maximal ideal and \( (N: ex) \supset (N: x) \supset (N: e) \supset (N: ex) \). Thus
\[(1-e) \in (N: e) = (N: x) \] and \[x - xe \in N. \] Since \[e x - e \in N, \] this implies that \[x - e \in N. \] Now let \[g = g^2 \] in \(R\) such that \[xg \in N. \] Since \[e - x \in N, \] \[eg \in N. \] Let \[e' = e - eg. \] Then \[g \cdot e' = 0 \] and \[e' \cdot e' = (e - eg)(e - eg) = e - eg - eg + eg = e'. \] \[e' + N = e + N = x + N. \] It is well known that if \(R/N\) is a semisimple ring with the minimum condition then \[1 + N = (x_1 + N) + (x_2 + N) + \cdots + (x_n + N) \] for some positive integer \(n\) where \[x_i - x_i^2 \in N, \ i = 1, 2, \cdots, n, \] \[x_i x_j \in N \] if \(i \neq j\) and \[(N: x_i), \] for each \(i, \) is a maximal right ideal (see [2, p. 46 and p. 50]). By the above argument, we can choose an orthogonal set of idempotents \(e_1, e_2, \cdots, e_n\) in \(R\) such that \[x_i - e_i \in N, \ i = 1, 2, \cdots, n, \] and \[1 + N = (e_1 + N) + (e_2 + N) + \cdots + (e_n + N). \] Now let \(y + N\) be an arbitrary nonzero idempotent in \(R/N. \) Then \[y + N = (e_1 y + N) + (e_2 y + N) + \cdots + (e_n y + N) \] and \[e_i y \cdot e_j y \in N \] if \(i \neq j\) and \[(N: e_i y) \] is a maximal ideal for all \(i\) such that \(e_i y \in N. \) There is an orthogonal set of idempotents \(a_1, a_2, \cdots, a_n\) in \(R\) such that \[y - (a_1 + a_2 + \cdots + a_n) \in N \] and \[(a_1 + a_2 + \cdots + a_n)^2 = (a_1 + a_2 + \cdots + a_n). \]

References


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