ON SOME CHARACTERISTIC PROPERTIES
OF SELF-INJECTIVE RINGS

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A ring with unit element is said to be left self-injective if and only if every (left) \( R \)-homomorphism of a left ideal of \( R \) into \( R \) can be given by the right multiplication of an element of \( R \). In [2], Ikeda-Nakayama introduced the following conditions in a ring \( R \) with unit element:

\((A)\) Every (left) \( R \)-homomorphism of a principal left ideal of \( R \) into \( R \) may be given by the right multiplication of an element of \( R \).

\((A_0)\) Every (left) \( R \)-homomorphism of a principal left ideal \( L \) of \( R \) into a residue module \( R / L' \), of \( R \) modulo a left ideal \( L' \), may be obtained by the right multiplication of an element, say \( c \), of \( R \): \( x \rightarrow xc \pmod{L'} \), \((x \in L)\).

\((B)\) If \( I \) is a finitely generated right ideal in \( R \), then the set of right annihilators of the set of left annihilators of \( I \) is \( I \).

\((B^*)\) If \( I \) is a principal right ideal in \( R \), then the set of right annihilators of the set of left annihilators of \( I \) is \( I \).

We introduce another condition:

\((C)\) If \( F \) is a finitely generated left free \( R \)-module and \( M \) is a cyclic submodule of \( F \) then any \( R \)-homomorphism of \( M \) into \( R \) can be extended to a \( R \)-homomorphism of \( F \) into \( R \).

In this paper, we shall prove the following: In a ring with 1, (B) holds if and only if (C) holds. If \( R \) is a ring with 1 such that every principal left ideal is projective, then the three conditions \((A)\), \((A_0)\) and \((B)\) are equivalent. If \( R \) is a ring with 1 such that the right singular ideal (refer to [4] for definition) is zero, then \( R \) is a semisimple ring with minimum conditions on one-sided ideals if and only if \( R \) satisfies the maximum condition for annihilator right ideals and the condition \((B)\). In particular, a regular ring \( R \) with 1 is a semisimple ring with minimum conditions on one-sided ideals if and only if it satisfies the maximum condition for annihilator right ideals. In a simple ring \( R \) with 1, the condition \((B^*)\) and the existence of a maximal annihilator left ideal in \( R \) are necessary and sufficient conditions for \( R \) to satisfy minimum conditions on one-sided ideals. In a ring with 1, the condition \((B^*)\) implies that the left singular ideal of \( R \) is, indeed, the Jacobson radical of \( R \).

In the sequel, if \( X \) is a subset in \( R \), we denote the set of left (right)

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annihilators of $X$ in $R$ by $l(X)(\gamma(X))$. In case $X = \{a\}$, one element set, then we let $l(X) = l(a)(\gamma(X) = \gamma(a))$.

**Theorem 1.** If $R$ is a ring with 1, then the condition (B) holds in $R$ if and only if the condition (C) holds in $R$.

**Proof.** Assume (B). Let

$$F = R \oplus R \oplus \cdots \oplus R \quad (n \text{ copies})$$

for some positive integer $n$ and let $M$ be a cyclic submodule of $F$. Then $M = Rm_0$ for some $m_0 \in F$ and $m_0 = a_1 + a_2 + \cdots + a_n$ for some $a_1$, $a_2$, \ldots, $a_n$ in $R$. Let $h$ be a $R$-homomorphism of $M$ into $R$. Then $h(m_0) = b$ for some $b \in R$. Let $L = \cap_{i=1}^n l(a_i)$. Then $L \subseteq l(b)$. Since $L = l(\sum_{i=1}^n a_i R)$, $\sum_{i=1}^n a_i R = r(L) \supseteq r(l(b))$ by (B). Hence $b = a_1 s_1 + a_2 s_2 + \cdots + a_n s_n$ for some $s_1$, $s_2$, \ldots, $s_n$ in $R$. Define $\bar{h}(r_1 + r_2 + \cdots + r_n) = \sum_{i=1}^n r_i s_i$ for any $r_1$, $r_2$, \ldots, $r_n$ in $R$. Then $\bar{h}$ is an $R$-homomorphism of $F$ into $R$ and $\bar{h}(m) = h(m)$ for all $m \in M$.

Assume (C). Let $I$ be a finitely generated right ideal of $R$, say $I = \sum_{i=1}^n x_i R$. Since $\gamma(l(I))$ always, it suffices to prove $\gamma(l(I)) \subseteq I$. For each $x \in \gamma(l(I))$, $(\cap_{i=1}^n l(x_i)) \cdot x = 0$. Define a (left) $R$-homomorphism $f$ from $R$ into the free left $R$-module

$$F = R \oplus R \oplus \cdots \oplus R \quad (n \text{ copies})$$

by $f(a) = ax_1 + ax_2 + \cdots + ax_n$ for all $a \in R$. Let $M = f(R)$. Define the $R$-homomorphism $g$ from $M$ into $R$ by $g(f(a)) = ax$ for all $a \in R$. We need to show that $g$ is indeed well defined. If $f(a) = f(b)$, $a$, $b \in R$, then $a - b \in \cap_{i=1}^n l(x_i)$ since

$$f(a) = ax_1 + ax_2 + \cdots + ax_n = bx_1 + bx_2 + \cdots + bx_n = f(b).$$

Hence $a - b \in l(x)$ and $ax = bx$. Since $M$ is a cyclic submodule of $F$, by (C) we may extend $g$ to $\bar{g}$ which is an $R$-homomorphism of $F$ into $R$. Now

$$x = \bar{g}(1) = \bar{g}(x_1 + x_2 + \cdots + x_n) = \bar{g}(x_1 + 0 + 0 + \cdots + 0) + \bar{g}(0 + x_2 + 0 + \cdots + 0) + \cdots + \bar{g}(0 + 0 + \cdots + 0 + x_n) = x_1 \bar{g}(1 + 0 + 0 + \cdots + 0) + x_2 \bar{g}(0 + 1 + 0 + \cdots + 0) + \cdots + x_n \bar{g}(0 + 0 + \cdots + 0 + 1).$$

Thus $x \in I$.

**Theorem 2.** If $R$ is a ring with 1 such that every principal left ideal is projective then the three conditions (A), (A) and (B) are equivalent.
Proof. Since (B) implies (A) always in any ring with 1 by [2, (i) of Theorem 1], we shall prove that (A)⇒(A₀)⇒(B). Let a be a nonzero element of R. Then Ra is projective. Hence the following diagram,

\[
\begin{array}{ccc}
R & \to & Ra \\
\downarrow & & \downarrow h \\
R & \to & Ra \\
\end{array}
\]

where \(i\) is the identity mapping, \(\pi_a(x) = xa\) for all \(x \in R\) and \(h\) is an \(R\)-homomorphism of \(Ra\) into \(R\), is commutative. By (A), \(h(a) = ar_0\) for some \(r_0\) in \(R\). Now \(ar_0a = h(a)a = \pi_a h(a) = a\). Thus \(R\) is a regular ring and by [2, Theorem 3], (A)⇒(A₀). Since (A₀) implies that \(R\) is a regular ring by [2, Theorem 3], and any finitely generated right ideal in a regular ring is a principal right ideal generated by an idempotent element (see, for example, [6, Lemma 15, p. 710]), (A₀) implies (B).

Lemma. Let \(R\) be a ring with unit element such that if \(I\) is a maximal right ideal in \(R\) then there is a nonzero right ideal \(K\) in \(R\) such that \(IK = (0)\). Then \(R\) is a semisimple ring with minimum conditions on one-sided ideals.

Proof. Let \(F\) be the right socle of \(R\). If \(1 \notin F\), then by Zorn's Lemma there exists a maximal right ideal, say \(I\) of \(R\) such that \(I \supseteq F\). Let \(K\) be a nonzero right ideal of \(R\) such that \(IK = (0)\). Then \(K\) must be a minimal right ideal of \(R\) since \(I\) is a maximal right ideal. Hence \(K\) is a minimal right ideal which is not contained in \(F\). This is impossible. Thus \(1 \in F\) and \(F = R\). From [1, Theorem 11, p. 61], the assertion follows.

We say a ring \(R\) satisfies the maximum condition for annihilator right ideals if every nonvacuous collection of annihilator right ideals of \(R\) contains a maximal element.

Theorem 3. Let \(R\) be a ring with unit element such that the right singular ideal of \(R\) is zero. Then the following two statements are equivalent:

(a) \(R\) is a semisimple ring with minimum conditions on one-sided ideals.

(b) \(R\) satisfies the maximum condition for annihilator right ideals and the condition (B).

Proof. Assume (a). Since any semisimple ring \(R\) with minimum conditions on one-sided ideals satisfies the maximum condition for annihilator right ideals and the condition (B), we have (b).
condition on right ideals satisfies the maximum condition for right ideals (see, for example, [3, p. 64]), \( R \) satisfies the maximum condition for annihilator right ideals. By [3, Structure Theorem (3), p. 12] and [2, (iii) of Theorem 1], \( R \) satisfies the condition (B). Conversely, assume (b). Let \( I \) be a maximal right ideal of \( R \). We shall prove that \( l(I) \neq 0 \). Let \( S \) be a family of annihilator left ideals in \( R \) such that \( S \) if and only if \( L = \bigcap_{i=1}^{n} l(x_i) \) for some finite number of \( x_i \) in \( I \). Since the maximum condition on annihilator right ideals implies the minimum condition on annihilator left ideals, we may choose a minimal member, say \( L_0 \in S \). Let \( L_0 = \bigcap_{j=1}^{k} l(x_j) \) for some \( x_1, x_2, \ldots, x_j \in I \). Since \( \bigcap_{j=1}^{k} l(x_j) = l(x_1R + x_2R + \cdots + x_jR) \), by (B) \( \gamma(L_0) = \gamma(l(x_1R + x_2R + \cdots + x_jR)) = x_1R + x_2R + \cdots + x_jR \subseteq I \). Hence \( L_0 \neq 0 \). If \( x \in I \) then \( l(x) \cap L_0 \subseteq L_0 \). Hence \( l(x) \cap L_0 = L_0 \) since \( L_0 \) is a minimal member of \( S \) and \( l(x) \cap L_0 \subseteq S \). Thus \( 0 \neq L_0 \subseteq l(x) \) and \( 0 \neq L_0 \subseteq \bigcap_{x \in I} l(x) = l(I) \). Since the right singular ideal of \( R \) is zero and \( l(I) \neq 0 \), there must exist a nonzero right ideal \( K \) in \( R \) such that \( I \cap K = 0 \). Thus, by the Lemma, (a) is true.

**Corollary.** If \( R \) is a regular ring with 1 such that it satisfies the maximum condition for annihilator right ideals, then \( R \) is a semisimple ring with minimum conditions on one-sided ideals.

**Proof.** As noted before, any finitely generated right ideal in a regular ring is a principal right ideal generated by an idempotent element. Hence the condition (B) is satisfied. By [5, p. 1386] a regular ring has zero right singular ideal. Thus by Theorem 3, the assertion is true.

**Theorem 4.** Let \( R \) be a simple ring with unit element. Then the following two statements are equivalent:

(a) There exists a maximal annihilator left ideal in \( R \) and \((B^*)\) holds in \( R \).

(b) \( R \) satisfies the minimum condition on one-sided ideals.

**Proof.** Clearly (b) implies (a). Assume (a). It is well known that a simple ring with a minimal one-sided ideal is isomorphic to a dense ring of linear transformations of finite rank of a vector space over a division ring. Hence it suffices to prove an existence of a minimal one-sided ideal in \( R \) in our case since \( 1 \in R \). We shall show that there is a maximal right ideal \( I \) in \( R \) which has zero intersection with some nonzero right ideal \( K \) in \( R \). Suppose that if \( I \) is a maximal right ideal in \( R \) then \( I \cap K \neq 0 \) for any nonzero right ideal \( K \) in \( R \). Let \( a \in R \), \( a \neq 0 \), such that \( l(a) \) is a maximal annihilator left ideal. Then \( a \cdot I \neq 0 \) since a simple ring with 1 has zero (right) singular ideal. Hence
Let \( x \in aI \cap I \) such that \( x \neq 0 \). Then \( x = ai \) for some \( i \in I \) and \( l(x) = l(ai) = l(a) \) since \( l(a) \subseteq l(x) \) and \( l(a) \) is a maximal annihilator left ideal. Now \( l(a) = l(x) = l(xR) \). Hence \( a \in \gamma(l(a)) = \gamma(l(xR)) = xR \subseteq I \). Thus \( a \) is contained in the intersection of all maximal right ideals \( I \) in \( R \). That is, the Jacobson radical of \( R \) is not zero. This is impossible.

**Theorem 5.** If \( R \) is a ring with 1 such that \((B^*)\) holds in \( R \) then the left singular ideal of \( R \) is, indeed, the Jacobson radical of \( R \).

**Proof.** Let \( x \) be a nonzero element in the left singular ideal of \( R \). Then \( 0 = l(1 - x) \). Otherwise, \( l(x) \cap l(1 - x) \neq 0 \) and if \( y \in l(x) \cap l(1 - x) \), \( y \neq 0 \), then \( y = xy = 0 \). Hence \( R = \gamma(l(1 - x)) = \gamma(l((1 - x)R)) = (1 - x)R \) by \((B^*)\). Thus every element of the left singular ideal of \( R \) is quasi-regular. Suppose there is an element \( a \) in the Jacobson radical of \( R \) which is not contained in the left singular ideal of \( R \). Then there is a nonzero left ideal \( K \) in \( R \) such that \( K \cap l(a) = 0 \). Let \( k \in K \) and \( k \neq 0 \). Then \( l(k) = l(ka) \) since \( xka = 0 \) if and only if \( xk = 0 \) for any \( x \in R \). Since \( l(kR) = l(k) = l(ka) = l(kaR) \), by \((B^*)\) \( kR = r(l(k)) = r(l(ka)) = kaR \). Hence \( k = kar \) for some \( r \in R \) and \( k(1 - ar) = 0 \). However, \( ar \) is in the Jacobson radical of \( R \). Hence \( (1 - ar)x = 1 \) for some \( x \in R \). This implies that \( k(1 - ar)x = k = 0 \). This is absurd. Thus the left singular ideal of \( R \) must be the Jacobson radical of \( R \).

**Corollary.** If \( R \) is a ring with 1 such that \((A)\) holds in \( R \), then the left singular ideal of \( R \) is the Jacobson radical of \( R \).

**Proof.** By [2, (i) of Theorem 1], \((A)\) is equivalent to \((B^*)\). Hence from Theorem 5, the assertion follows.

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**References**


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