COMMUTATIVITY OF THE INvariant DIFFERENTIAL OPERATORS ON A SYMMETRIC SPACE

WILLIAM SMOKE

It is known (see Helgason [1]) that the algebra of invariant differential operators on a Riemannian symmetric space is commutative. The algebra may be defined algebraically. We give here an algebraic proof of its commutativity.

Let \( g \) be a Lie algebra over a field of characteristic zero and let \( \mathfrak{f} \) be a subalgebra of \( g \). Extend the adjoint representation of \( \mathfrak{f} \) on \( g \) to the universal enveloping algebra \( U(g) \) of \( g \) so that \( \text{ad} \, x \), \( x \in \mathfrak{f} \), acts as a derivation of \( U(g) \). Then \( (\text{ad} \, x)(u) = xu - ux \) for \( u \in U(g) \). For this is the case if \( u \) belongs to \( g \), and \( g \) generates \( U(g) \). It follows from the formula that the invariant elements of \( U(g) \)—those annihilated by the action of \( \mathfrak{f} \)—are the elements of \( U(g) \) which commute with the elements of \( \mathfrak{f} \). Let \( U(g)^{\mathfrak{f}} \) be the subalgebra of invariant elements.

Now let \( U(g)^{\mathfrak{f}} \) be the left ideal generated by \( \mathfrak{f} \) in \( U(g) \). This left ideal is preserved by the action of \( \mathfrak{f} \). Moreover, \( [U(g)^{\mathfrak{f}}]^{\mathfrak{f}} = U(g)^{\mathfrak{f}} \) \( \cap U(g)^{\mathfrak{f}} \) is a two-sided ideal in \( U(g)^{\mathfrak{f}} \). Let \( U(\mathfrak{g}, \mathfrak{f}) \) be the quotient algebra. If \( \mathfrak{g} \) is the Lie algebra of a Lie group \( G \) and \( \mathfrak{f} \) that of a connected subgroup \( K \) then \( U(\mathfrak{g}, \mathfrak{f}) \) is the algebra of invariant differential operators on the homogeneous space \( G/K \) (Smoke [2]). If \( K \) is compact and the fixed point set of an involutive automorphism of \( G \) then \( G/K \) is a Riemannian symmetric space and the algebra \( U(\mathfrak{g}, \mathfrak{f}) \) is commutative.

**Theorem.** Suppose that \( \mathfrak{g} \) is equipped with an involutive Lie algebra automorphism \( z \mapsto \bar{z} \) and let \( \mathfrak{f} \) be the subalgebra of fixed elements. Suppose that \( \mathfrak{f} \) is reductive in \( \mathfrak{g} \). Then \( U(\mathfrak{g}, \mathfrak{f}) \) is commutative.

The proof essentially reduces to the fact than an automorphism and an antiautomorphism coincide on \( U(\mathfrak{g}, \mathfrak{f}) \).

Extend the Lie algebra automorphism \( z \mapsto \bar{z} \) to an automorphism \( u \mapsto \bar{u} \) of the algebra \( U(\mathfrak{g}) \). Then for \( x \in \mathfrak{f} \), \( (\text{ad} \, x)(\bar{u}) = x\bar{u} - \bar{u}x = (xu)^- - (ux)^- = ( (\text{ad} \, x)(u) )^- \), so \( u \mapsto \bar{u} \) preserves the subalgebra \( U(\mathfrak{g})^{\mathfrak{f}} \). The ideal \( U(\mathfrak{g})^{\mathfrak{f}} \) is also preserved, so \( [U(\mathfrak{g})^{\mathfrak{f}}]^{\mathfrak{f}} \) is preserved and an automorphism is induced on \( U(\mathfrak{g}, \mathfrak{f}) \).

Recall that \( U(\mathfrak{g}) \) has a canonical antiautomorphism \( u \mapsto u^* \), characterized by the fact that \( z^* = -z \) for \( z \in \mathfrak{g} \). We have \( (\text{ad} \, x)(u^*) \)
= xu* − u*x = (xu − ux)* = [(ad x)(u)]* for x ∈ ℱ and u ∈ U(𝔤), so the subalgebra U(𝔤)ant is preserved. Evidently the left ideal U(𝔤)ant is mapped into the right ideal ℱU(𝔤). Thus [U(𝔤)ant]ant is mapped into [ℓU(𝔤)]ant. We must show that [ℓU(𝔤)]ant is contained in [U(𝔤)ant]ant. Since ℱ is reductive in ℱ the representation of ℱ on U(𝔤) is semisimple and U(𝔤) is the direct sum U(𝔤) = U(𝔤)ant ⊕ [ℓ, U(𝔤)]. Here, [ℓ, U(𝔤)] is the subspace generated by the commutators xu − ux, x ∈ ℱ and u ∈ U(𝔤). But ℱU(𝔤) ⊂ U(𝔤)ant + [ℓ, U(𝔤)], and it follows that the projection of ℱU(𝔤) on U(𝔤)ant is contained in the projection of U(𝔤)ant. Since ℱU(𝔤) and U(𝔤)ant are invariant subspaces of U(𝔤) their projections on U(𝔤)ant are respectively their intersections with this subspace. This shows that [ℓU(𝔤)]ant is contained in [U(𝔤)ant]ant. The antiautomorphism u → u* therefore preserves the ideal [U(𝔤)ant]ant, inducing an antiautomorphism on U(𝔤)ant.

It remains to show that the induced automorphism and antiautomorphism coincide. For this, we find a subspace of U(𝔤)ant, mapping onto U(𝔤)ant, on which the automorphism and antiautomorphism of U(𝔤) coincide.

Recall that U(𝔤) is filtered by subspaces Up(𝔤), where Up(𝔤) is spanned by the monomials of degree at most p in the elements of ℱ. The associated graded algebra is the symmetric algebra S(𝔤) of ℱ. If Sp(𝔤) is the component of degree p, the natural map from Up(𝔤) onto Sp(𝔤) has kernel Up−1(𝔤) and takes a monomial z1 · · ·zp of degree p into the corresponding commutative monomial z1 · · ·zp in Sp(𝔤). There is a linear map λ: S(𝔤) → U(𝔤) defined by

λ(z1 · · ·zp) = \frac{1}{p!} \sum \sigma z_{\sigma 1} · · · z_{\sigma p},

the sum running over all permutations. It is clear that λ: Sp(𝔤) → Up(𝔤) followed by the natural map Up(𝔤) → Sp(𝔤) is the identity on Sp(𝔤). This implies that the element z1 · · ·zp − λ(z1 · · ·zp) of Up(𝔤) belongs to Up−1(𝔤).

Now let b be the subspace of those elements y of ℱ which satisfy jy = −y. Then ℱ = ℱ ⊕ b. We may regard S(b) as a subalgebra of S(𝔤).

**Lemma.** U(𝔤)ant + λ[S(b)] = U(𝔤).

To prove the lemma, choose an ordered basis (x_i) for ℱ and ordered basis (y_j) for b. The Poincare-Birkhoff-Witt theorem then implies that the monomials y_{j_1} · · · y_{j_p}x_{i_1} · · · x_{i_q}, j_1 ≤ · · · ≤ j_p and i_1 ≤ · · · ≤ i_q, form a basis of U(𝔤). Clearly U(𝔤)ant contains all but those of the form y_{j_1} · · · y_{j_p}. The ground field U_0(𝔤) belongs to λ[S(b)]. Assume that U_{p−1}(𝔤) is contained in U(𝔤)ant + λ[S(b)]. Since y_{j_1} · · · y_{j_p}
\[ -\lambda(y_{j_1} \cdots y_{j_p}) \text{ belongs to } U_{p-1}(g) \text{ we find that } U_p(g) \text{ is contained in } U(g)\mathfrak{f} + \lambda[S(b)], \text{ so the lemma follows inductively.} \]

The action of \( \mathfrak{f} \) on \( g \) extends to \( S(g) \), with \( \text{ad} \, x, \, x \in \mathfrak{f} \), again acting as a derivation. Since \( b \) is invariant in \( g \), \( S(b) \) is invariant in \( S(g) \). It is easy to see that \( \lambda: S(g) \rightarrow U(g) \) commutes with the action of \( \mathfrak{f} \). It follows that \( \lambda[S(b)] \) is an invariant subspace of \( U(g) \). Each of the spaces in the lemma decomposes as a direct sum under the action of \( \mathfrak{f} \) and \( U(g)^\mathfrak{f} = [U(g)\mathfrak{f}]^\mathfrak{f} + \lambda[S(b)]^\mathfrak{f} \). It follows that the natural map \( U(g)^\mathfrak{f} \rightarrow U(g, \mathfrak{f}) \) maps \( \lambda[S(b)]^\mathfrak{f} \) onto \( U(g, \mathfrak{f}) \).

It remains only to show that the automorphism and antiautomorphism of \( U(g) \) agree on \( \lambda[S(b)]^\mathfrak{f} \). In fact they agree on \( \lambda[S(b)] \). For if \( y_1, \cdots, y_p \) are any elements of \( b \) we have \( \lambda(y_1 \cdots y_p)^- = (-1)^p \lambda(y_1 \cdots y_p) \). On the other hand, \( \lambda(y_1 \cdots y_p)^* = (-1)^p \lambda(y_1 \cdots y_p) \) also, since \( \lambda(y_1 \cdots y_p) \) is a sum over all permutations.

It is now clear that \( U(g, \mathfrak{f}) \) is commutative. If \( v \) and \( w \) belong to \( U(g, \mathfrak{f}) \) then \( (vw)^- = (vw)^* = w^*v^* = \tilde{w} \tilde{v} = (\tilde{wv})^- \), so \( vw = wv \).

**References**


**University of California, Irvine**