COMMUTATIVITY OF THE INVARIANT DIFFERENTIAL OPERATORS ON A SYMMETRIC SPACE[1]

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It is known (see Helgason [1]) that the algebra of invariant differential operators on a Riemannian symmetric space is commutative. The algebra may be defined algebraically. We give here an algebraic proof of its commutativity.

Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic zero and let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$. Extend the adjoint representation of $\mathfrak{k}$ on $\mathfrak{g}$ to the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ so that $\operatorname{ad} x$, $x \in \mathfrak{k}$, acts as a derivation of $U(\mathfrak{g})$. Then $(\operatorname{ad} x)(u) = xu - ux$ for $u \in U(\mathfrak{g})$. For this is the case if u belongs to $\mathfrak{g}$, and $\mathfrak{g}$ generates $U(\mathfrak{g})$. It follows from the formula that the invariant elements of $U(\mathfrak{g})$—those annihilated by the action of $\mathfrak{k}$—are the elements of $U(\mathfrak{g})$ which commute with the elements of $\mathfrak{k}$. Let $U(\mathfrak{g})^{\mathfrak{k}}$ be the subalgebra of invariant elements.

Now let $U(\mathfrak{g})\mathfrak{k}$ be the left ideal generated by $\mathfrak{k}$ in $U(\mathfrak{g})$. This left ideal is preserved by the action of $\mathfrak{k}$. Moreover, $[U(\mathfrak{g})\mathfrak{k}]^{\mathfrak{k}} = U(\mathfrak{g})\mathfrak{k} \cap U(\mathfrak{g})^{\mathfrak{k}}$ is a two-sided ideal in $U(\mathfrak{g})^{\mathfrak{k}}$. Let $U(\mathfrak{g}, \mathfrak{k})$ be the quotient algebra. If $\mathfrak{g}$ is the Lie algebra of a Lie group G and $\mathfrak{k}$ that of a connected subgroup K then $U(\mathfrak{g}, \mathfrak{k})$ is the algebra of invariant differential operators on the homogeneous space G/K (Smoke [2]). If K is compact and the fixed point set of an involutive automorphism of G then G/K is a Riemannian symmetric space and the algebra $U(\mathfrak{g}, \mathfrak{k})$ is commutative.

THEOREM. *Suppose that $\mathfrak{g}$ is equipped with an involutive Lie algebra automorphism $z \rightarrow \bar{z}$ and let $\mathfrak{k}$ be the subalgebra of fixed elements. Suppose that $\mathfrak{k}$ is reductive in $\mathfrak{g}$. Then $U(\mathfrak{g}, \mathfrak{k})$ is commutative.*

The proof essentially reduces to the fact than an automorphism and an antiautomorphism coincide on $U(\mathfrak{g}, \mathfrak{k})$.

Extend the Lie algebra automorphism $z \rightarrow \bar{z}$ to an automorphism $u \rightarrow \bar{u}$ of the algebra $U(\mathfrak{g})$. Then for $x \in \mathfrak{k}$, $(\operatorname{ad} x)(\bar{u}) = x\bar{u} - \bar{u}x = (xu)^- - (ux)^- = ((\operatorname{ad} x)(u))^-$, so $u \rightarrow \bar{u}$ preserves the subalgebra $U(\mathfrak{g})^{\mathfrak{k}}$. The ideal $U(\mathfrak{g})\mathfrak{k}$ is also preserved, so $[U(\mathfrak{g})\mathfrak{k}]^{\mathfrak{k}}$ is preserved and an automorphism is induced on $U(\mathfrak{g}, \mathfrak{k})$.

Recall that $U(\mathfrak{g})$ has a canonical antiautomorphism $u \rightarrow u^*$, characterized by the fact that $z^* = -z$ for $z \in \mathfrak{g}$. We have $(\operatorname{ad} x)(u^*)$

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$=xu^*-u^*x=(xu-ux)^*=[(\mathrm{ad}\,x)(u)]^*$ for $x\in\mathfrak{k}$ and $u\in U(\mathfrak{g})$, so the subalgebra $U(\mathfrak{g})^{\mathfrak{k}}$ is preserved. Evidently the left ideal $U(\mathfrak{g})\mathfrak{k}$ is mapped into the right ideal $\mathfrak{k}U(\mathfrak{g})$. Thus $[U(\mathfrak{g})\mathfrak{k}]^{\mathfrak{k}}$ is mapped into $[\mathfrak{k}U(\mathfrak{g})]^{\mathfrak{k}}$. We must show that $[\mathfrak{k}U(\mathfrak{g})]^{\mathfrak{k}}$ is contained in $[U(\mathfrak{g})\mathfrak{k}]^{\mathfrak{k}}$. Since $\mathfrak{k}$ is reductive in $\mathfrak{g}$ the representation of $\mathfrak{k}$ on $U(\mathfrak{g})$ is semisimple and $U(\mathfrak{g})$ is the direct sum $U(\mathfrak{g})=U(\mathfrak{g})^{\mathfrak{k}}\oplus[\mathfrak{k}, U(\mathfrak{g})]$. Here, $[\mathfrak{k}, U(\mathfrak{g})]$ is the subspace generated by the commutators $xu-ux$, $x\in\mathfrak{k}$ and $u\in U(\mathfrak{g})$. But $\mathfrak{k}U(\mathfrak{g})\subset U(\mathfrak{g})\mathfrak{k}+[\mathfrak{k}, U(\mathfrak{g})]$, and it follows that the projection of $\mathfrak{k}U(\mathfrak{g})$ on $U(\mathfrak{g})^{\mathfrak{k}}$ is contained in the projection of $U(\mathfrak{g})\mathfrak{k}$. Since $\mathfrak{k}U(\mathfrak{g})$ and $U(\mathfrak{g})\mathfrak{k}$ are invariant subspaces of $U(\mathfrak{g})$ their projections on $U(\mathfrak{g})^{\mathfrak{k}}$ are respectively their intersections with this subspace. This shows that $[\mathfrak{k}U(\mathfrak{g})]^{\mathfrak{k}}$ is contained in $[U(\mathfrak{g})\mathfrak{k}]^{\mathfrak{k}}$. The antiautomorphism $u\rightarrow u^*$ therefore preserves the ideal $[U(\mathfrak{g})\mathfrak{k}]^{\mathfrak{k}}$, inducing an antiautomorphism on $U(\mathfrak{g}, \mathfrak{k})$.

It remains to show that the induced automorphism and antiautomorphism coincide. For this, we find a subspace of $U(\mathfrak{g})^{\mathfrak{k}}$, mapping onto $U(\mathfrak{g}, \mathfrak{k})$, on which the automorphism and antiautomorphism of $U(\mathfrak{g})$ coincide.

Recall that $U(\mathfrak{g})$ is filtered by subspaces $U_p(\mathfrak{g})$, where $U_p(\mathfrak{g})$ is spanned by the monomials of degree at most p in the elements of $\mathfrak{g}$. The associated graded algebra is the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$. If $S^p(\mathfrak{g})$ is the component of degree p, the natural map from $U_p(\mathfrak{g})$ onto $S^p(\mathfrak{g})$ has kernel $U_{p-1}(\mathfrak{g})$ and takes a monomial $z_1\cdots z_p$ of degree p into the corresponding commutative monomial $z_1\cdots z_p$ in $S^p(\mathfrak{g})$. There is a linear map $\lambda\colon S(\mathfrak{g})\rightarrow U(\mathfrak{g})$ defined by

$$\lambda(z_1\cdots z_p)=\frac{1}{p!}\sum_{\sigma} z_{\sigma 1}\cdots z_{\sigma p},$$

the sum running over all permutations. It is clear that $\lambda\colon S^p(\mathfrak{g})\rightarrow U_p(\mathfrak{g})$ followed by the natural map $U_p(\mathfrak{g})\rightarrow S^p(\mathfrak{g})$ is the identity on $S^p(\mathfrak{g})$. This implies that the element $z_1\cdots z_p-\lambda(z_1\cdots z_p)$ of $U_p(\mathfrak{g})$ belongs to $U_{p-1}(\mathfrak{g})$.

Now let $\mathfrak{b}$ be the subspace of those elements y of $\mathfrak{g}$ which satisfy $\bar{y}=-y$. Then $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{b}$. We may regard $S(\mathfrak{b})$ as a subalgebra of $S(\mathfrak{g})$.

LEMMA. $U(\mathfrak{g})\mathfrak{k}+\lambda[S(\mathfrak{b})] = U(\mathfrak{g})$.

To prove the lemma, choose an ordered basis (x_i) for $\mathfrak{k}$ and ordered basis (y_j) for $\mathfrak{b}$. The Poincare-Birkhoff-Witt theorem then implies that the monomials $y_{j_1}\cdots y_{j_p}x_{i_1}\cdots x_{i_q}$, $j_1\leqq\cdots\leqq j_p$ and $i_1\leqq\cdots\leqq i_q$, form a basis of $U(\mathfrak{g})$. Clearly $U(\mathfrak{g})\mathfrak{k}$ contains all but those of the form $y_{j_1}\cdots y_{j_p}$. The ground field $U_0(\mathfrak{g})$ belongs to $\lambda[S(\mathfrak{b})]$. Assume that $U_{p-1}(\mathfrak{g})$ is contained in $U(\mathfrak{g})\mathfrak{k}+\lambda[S(\mathfrak{b})]$. Since $y_{j_1}\cdots y_{j_p}$

$-\lambda(y_{j_i} \cdots y_{j_p})$ belongs to $U_{p-1}(\mathfrak{g})$ we find that $U_p(\mathfrak{g})$ is contained in $U(\mathfrak{g})\mathfrak{k}+\lambda[S(\mathfrak{b})]$, so the lemma follows inductively.

The action of $\mathfrak{k}$ on $\mathfrak{g}$ extends to $S(\mathfrak{g})$, with ad x, $x\in\mathfrak{k}$, again acting as a derivation. Since $\mathfrak{b}$ is invariant in $\mathfrak{g}$, $S(\mathfrak{b})$ is invariant in $S(\mathfrak{g})$. It is easy to see that λ: $S(\mathfrak{g})\rightarrow U(\mathfrak{g})$ commutes with the action of $\mathfrak{k}$. It follows that $\lambda[S(\mathfrak{b})]$ is an invariant subspace of $U(\mathfrak{g})$. Each of the spaces in the lemma decomposes as a direct sum under the action of $\mathfrak{k}$ and $U(\mathfrak{g})^{\mathfrak{k}}=[U(\mathfrak{g})\mathfrak{k}]^{\mathfrak{k}}+\lambda[S(\mathfrak{b})]^{\mathfrak{k}}$. It follows that the natural map $U(\mathfrak{g})^{\mathfrak{k}}\rightarrow U(\mathfrak{g}, \mathfrak{k})$ maps $\lambda[S(\mathfrak{b})]^{k}$ onto $U(\mathfrak{g}, \mathfrak{k})$.

It remains only to show that the automorphism and antiautomorphism of $U(\mathfrak{g})$ agree on $\lambda[S(\mathfrak{b})]^{\mathfrak{k}}$. In fact they agree on $\lambda[S(\mathfrak{b})]$. For if $y_1, \cdots, y_p$ are any elements of $\mathfrak{b}$ we have $\lambda(y_1 \cdots y_p)^- =(-1)^p\lambda(y_1 \cdots y_p)$. On the other hand, $\lambda(y_1 \cdots y_p)^*=(-1)^p \cdot\lambda(y_1 \cdots y_p)$ also, since $\lambda(y_1 \cdots y_p)$ is a sum over all permutations.

It is now clear that $U(\mathfrak{g}, \mathfrak{k})$ is commutative. If v and w belong to $U(\mathfrak{g}, \mathfrak{k})$ then $(vw)^-=(vw)^*=w^*v^*=\bar{w}\bar{v}=(wv)^-$, so $vw=wv$.

References

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2. W. Smoke, *Invariant differential operators*, Trans. Amer. Math. Soc. **127** (1967), 460–494.

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