THE DISTANCE FROM \( U(z) \cdot H^p \) TO 1

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If \( U(z) \) is an inner function, then the set \( U(z) \cdot H^p \) of all \( H^p \) multiples of \( U(z) \) forms a closed subspace of \( H^p \). In this note we compute the \( H^p \) distance between the constant function 1 and this closed subspace. It is of course well known that this distance is 0 if and only if \( U(z) \) is a constant, i.e. if and only if \( |U(0)| = 1 \). We will prove

**Theorem.** \( \text{dist} (1, U(z) \cdot H^p) = (1 - |U(0)|^2)^{1/p}, \quad p \geq 1. \)

**Proof.** With

\[
f_p(z) = \frac{1 - (1 - U(z) \overline{U}(0))^{2/p}}{U(z)}
\]

we have

\[
\|1 - U(z)f_p(z)\| = \left( \frac{1}{2\pi} \int_0^{2\pi} |1 - U(z) \overline{U}(0)|^2d\theta \right)^{1/p}
\]

\[
= \left( \frac{1}{2\pi} \int_0^{2\pi} |U(z) - U(0)|^2d\theta \right)^{1/p} = (1 - |U(0)|^2)^{1/p},
\]

so that this distance is surely \( \leq (1 - |U(0)|^2)^{1/p} \) and we need only show that \( f_p(z) \) is the closest function to 1, i.e. that, for all \( f(z) \in H^p \),

\[
(1) \quad \|1 - U(z)f(z)\|_p \geq (1 - |U(0)|^2)^{1/p}.
\]

Consider

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\[ I = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{1}{U(z)} - f(z) \right) (U(z) - U(0))(1 - U(z)\overline{U(0)})^{1-2/p}d\theta \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{1}{U(z)} - f_p(z) \right) (U(z) - U(0))(1 - U(z)\overline{U(0)})^{1-2/p}d\theta \]

(since \( f(z) - f_p(z) \) is analytic).

Thus,
\[ I = \frac{1}{2\pi} \int_0^{2\pi} \frac{(U(z) - U(0))}{U(z)} (1 - U(z)\overline{U(0)})d\theta \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} |U(z) - U(0)|^2d\theta, \]

or

\[ I = 1 - |U(0)|^2. \]

On the other hand, by Hölder's inequality, for \( p > 1 \),
\[ I \leq \left( \frac{1}{2\pi} \int_0^{2\pi} |1 - U(z)f(z)|^p d\theta \right)^{1/p} \]
\[ \left( \frac{1}{2\pi} \int_0^{2\pi} |U(z) - U(0)|^{2-2/p}(p/(p-1))d\theta \right)^{(p-1)/p} \]
\[ = \|1 - U(z)f(z)\|_p \cdot \left( \frac{1}{2\pi} \int_0^{2\pi} |U(z) - U(0)|^2 d\theta \right)^{1-1/p}. \]

The same clearly holds for \( p = 1 \).

Hence

\[ I \leq \|1 - U(z)f(z)\|_p \cdot (1 - |U(0)|^2)^{1-1/p}. \]

Comparing (2) and (3) yields (1) immediately.

In the simple case of \( H^2 \) and a finite Blaschke product we obtain the

**Corollary.** Let \( |z_i| < 1, i = 1, 2, \ldots, N \). The minimum value of
\[ \sum\! |C_n|^2 \text{ subject to } \sum_{n=0}^\infty C_nz_i^n = 1, i = 1, 2, \ldots, N, \text{ is } 1 - |z_1z_2 \cdots z_N|^2. \]

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