LIMIT OF A SEQUENCE OF FUNCTIONS WITH ONLY COUNTABLY MANY POINTS OF DISCONTINUITY

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1. Introduction and statement of results.

1.1. Introduction. We present here first an approximation theorem (Theorem 1) for certain limit functions defined on a general topological space. This strengthens a result which may be found in Hausdorff [1]. With the aid of this theorem we characterize the limits of some classes of discontinuous functions in Theorem 2.

Denote by $S$ a topological space. All functions considered are real valued. Convergence means pointwise convergence unless otherwise stated. Suppose $f$ is a function defined on $S$, $x \in S$, and $f$ is not continuous at $x$. The statement that $(x, f(x))$ is a removable point of discontinuity means that there exists a function $g$ which agrees with $f$ on $S - \{x\}$ and which is continuous at $x$. The statement that the function $u$ defined on $S$ is upper semicontinuous means that, if $x \in S$ and $d > u(x)$, then there exists a neighborhood $V$ of $x$ such that, if $y \in V$, then $d > u(y)$. The function $l$ is lower semicontinuous if $-l$ is upper semicontinuous.

1.2. Statement of Theorems.

Theorem 1. Suppose $M$ is a linear space of real valued functions defined on $S$ which contains a nonzero constant function and which is closed under the operation of absolute value, and $U$ is the set to which $u$ belongs only in case $u$ is the greatest lower bound of a countable subset of $M$. Then, if the function $f$ defined on $S$ is the limit of a sequence of functions in $M$, it is the uniform limit of a sequence each term of which is the difference of two members of $U$, each of which is bounded above.

Theorem 2. Suppose $S$ is perfectly normal and $f$ is a function defined on $S$. Each two of the following three statements are equivalent:

1. the function $f$ is the limit of a sequence of functions, each of which has at most a finite number of points of discontinuity, each of which is removable;

2. the function $f$ is the limit of a sequence of functions, each of which has at most countably many points of discontinuity; and

3. there exist a function $g$ which is the limit of a sequence of continuous functions and a countable subset $T$ of $S$ such that, if $x \in S - T$, $f(x) = g(x)$.

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2. Proof of theorems.

2.1. Proof of Theorem 1. The following facts should be noted: As $M$ is closed under the operation of absolute value, if each of $h$ and $k$ is in $M$ then each of $\max\{h, k\}$ and $\min\{h, k\}$ is in $M$. Also, each member of $U$ is the limit from above of a monotonic sequence of members of $M$.

Suppose $\{f_p\}_{p=1}^{\infty}, f_p \in M$, converges to $f$. For each positive integer $p$ define $l_p = \inf \{f_p, f_{p+1}, \ldots \}$ and $u_p = \sup \{f_p, f_{p+1}, \ldots \}$. Thus $-l_p \in U, u_p \in U, l_p \leq f_p \leq u_p, l_{p+1} \leq l_p$, and $u_p \leq u_{p+1}$. Suppose $c > 0$. Define $R_p = \{x \mid l_p(x) - u_p(x) \leq c\}$. We assume that $R_1$ contains at least one point. Note that $\bigcup_{p=1}^{\infty} R_p = S$.

Now we show that the function $g_p$ defined such that $g_p(x) = 1$ if $x \in R_p$ and $g_p(x) = 0$ if $x \notin R_p$ is in $U$. There exists a monotonic sequence $\{v_n\}_{n=1}^{\infty}, v_n \in M$, converging from above to $u_p$ and there exists a monotonic sequence $\{w_n\}_{n=1}^{\infty}, w_n \in M$, converging from below to $l_p$. Define $h_n = w_n - v_n$ and $Q_n = \{x \mid h_n(x) \leq c\}$. Thus $R_p = \bigcap_{n=1}^{\infty} Q_n$. Define $q_n(x) = 1$ if $x \in Q_n$ and $q_n(x) = 0$ if $x \notin Q_n$. Define $d_n = \max\{1 - (\max\{h_n, c\} - c), 0\}$. Now $d_n(x) = 1$ if $x \in Q_n$ and $0 \leq d_n(x) < 1$ if $x \notin Q_n$. Define $r_{n,i} = \max\{i \cdot d_n - i + 1, 0\}$. Thus $\{r_{n,i}\}_{i=1}^{\infty}, r_{n,i} \in M$, is a monotonic sequence converging to $q_n$ from above. Define $\alpha_{i,p} = \min\{r_{1,i}, r_{2,i}, \ldots, r_{i,i}\}$. Then it is true that $\{\alpha_{i,p}\}_{i=1}^{\infty}, \alpha_{i,p} \in M$, is a monotonic sequence converging to $g_p$ from above and therefore $g_p \in U$.

Define $f_0 = 0,$

$$s_p = \sum_{n=1}^{p} \max\{f_n - f_{n-1}, 0\},$$

and

$$t_p = \sum_{n=1}^{p} \min\{f_n - f_{n-1}, 0\}.$$

Note that $s_p + t_p = f_p$. Define $h(x) = s_p(x)$ and $k(x) = t_p(x)$ if $x \in R_p$ but $x \notin R_{p-1}$. Note that both $k$ and $-h$ are bounded above.

Now we show that each of $k$ and $-h$ is in $U$. Define $\beta_{i,p} = i \cdot \alpha_{i,p} - i$. Thus $\beta_{i,p}(x) = 0$ for $x \in R_p, \beta_{i,p}(x) < 0$ for $x \notin R_p$, and $\beta_{i,p}(x) \to -\infty$ as $i \to \infty$ if $x \in R_p$. Define $\delta_{i,1} = \max\{\beta_{i,1} + t_1, t_2\}$. Define $\delta_{i,n} = \max\{\beta_{i,n} + \delta_{i,n-1}, t_{n+1}\}$ if $n > 1$. Define $\gamma_{i} = \delta_{i,i}$. If $p \leq i$ and $x \in R_p, \gamma_{i}(x) = \delta_{i,p}(x)$. Thus $\{\gamma_{i}\}_{i=1}^{\infty}, \gamma_{i} \in M$, is a monotonic sequence converging from above to $k$ and $k \in U$. A similar argument shows that $-h$ is in $U$.

If $x \in R_p$ but $x \notin R_{p-1}$, then $h(x) = s_p(x), k(x) = t_p(x)$, and $h(x) + k(x) = f_p(x)$. By the way $R_p$ was defined
\[ |f(x) - h(x) - k(x)| = |f(x) - f_p(x)| \leq c. \]

2.2. Notation. Suppose \( R \) is a subset of \( S \). Denote by \( U(R) \), \( L(R) \), and \( C(R) \), respectively, the set of all upper semicontinuous functions, lower semicontinuous functions, and continuous functions defined on \( R \). Denote by \( C_1(R) \) the set of all functions which are the limit of a sequence of members of \( C(R) \). Denote by \( U(R) + L(R) \) the set \( \{ f | f = h + k, h \in U(R), k \in L(R) \} \).

2.3. Some properties of semicontinuous functions. Certain facts concerning semicontinuous functions should be recalled. Property (1) is that if \( \{ f_p \}_{p=1}^{\infty}, f_p \subseteq C(S) \), converges to \( f \), then there exists a sequence \( \{ u_p \}_{p=1}^{\infty}, u_p \subseteq U(S), u_{p+1} \geq u_p \), converging to \( f \) and a sequence \( \{ l_p \}_{p=1}^{\infty}, l_p \subseteq L(S), l_{p+1} \leq l_p \), converging to \( f \) such that \( l_p \geq f_p \geq u_p \). This can be verified by defining \( u_p = \text{g.l.b.} \{ f_n | n = p, p + 1, \cdots \} \) and \( l_p = \text{l.u.b.} \{ f_n | n = p, p + 1, \cdots \} \). Property (2) is that \( S \) is perfectly normal if and only if every \( u \in U(S) \) is the limit from above of a monotonic sequence of continuous functions. This is a theorem due to Hing Tong [2].

2.4 Proof of Theorem 2.

(a) \( 1 \rightarrow 2 \): This follows immediately.

(b) \( 2 \rightarrow 3 \): Suppose \( \{ f_p \}_{p=1}^{\infty} \) is a sequence of functions defined on \( S \) converging to \( f \) such that each term of the sequence has at most countably many points of discontinuity. Define \( T = \{ x | f_p \text{ is discontinuous at } x \text{ for some positive integer } p \} \). The set \( T \) is countable. Define \( R = S - T, r_p \) to be the restriction of \( f_p \) to \( R \), and \( r \) to be the restriction of \( f \) to \( R \). Thus, \( \{ r_p \}_{p=1}^{\infty}, r_p \subseteq C(R) \), converges to \( r \). Now we apply Theorem 1. We take \( M \) to be the set of all continuous functions defined on \( R \). The limit from above of a monotonic sequence of continuous functions is upper semicontinuous. Therefore, the set \( U \) of Theorem 1 is a subset of \( U(R) \). Thus, \( r \) is the uniform limit of a sequence \( \{ b_p \}_{p=1}^{\infty} \), where each \( b_p \) is the difference of two upper semicontinuous functions, each bounded above. A function \( u \subseteq U(R) \) which is bounded above can be extended to a function \( v \subseteq U(S) \) by defining \( v(x) = \text{g.l.b.} \{ u(y) | y \in V \} \) for \( x \) where the lower bound exists. There exist at most a countable number of points \( x_1, x_2, \cdots \) of \( S \) where the lower bound does not exist. At these points define \( v(x_p) = -p \). By this means \( b_p \) can be extended to a function \( d_p \subseteq U(S) + L(S) \). Since \( \{ d_p \}_{p=1}^{\infty} \) converges uniformly to \( f \) on \( R \), we can assume without loss of generality that \( |d_p(x) - f(x)| < 1/p \) for \( x \in R \) and all \( p \). Define \( c_p = d_p + 1/p \) and \( e_p = d_p - 1/p \). If \( x \in R \) and \( p \) and \( j \) are positive integers, \( c_p(x) \geq f(x) \geq e_j(x) \). Also, if \( x \in R, c_p(x) \rightarrow f(x) \) as \( p \rightarrow \infty \) and \( e_p(x) \rightarrow f(x) \) as \( p \rightarrow \infty \).
Define \( s_p = \min \{ c_1, c_2, \ldots, c_p \} \) and \( t_p = \max \{ e_1, e_2, \ldots, e_p \} \). If \( x \in \mathbb{R} \), \( s_p(x) \leq f(x) \leq t_p(x) \). If \( x \in S \), \( s_p(x) \leq s_{p+1}(x) \) and \( t_p(x) \leq t_{p+1}(x) \).

Define \( v_p = \min \{ s_1, t_1 \}, \max \{ s_2, t_2 \}, \ldots, \max \{ s_p, t_p \} \) and \( w_p = \max \{ \min \{ s_1, t_1 \}, \min \{ s_2, t_2 \}, \ldots, \min \{ s_p, t_p \} \} \). If \( x \in \mathbb{R} \), \( v_p(x) \to f(x) \) as \( p \to \infty \) and \( w_p(x) \to f(x) \) as \( p \to \infty \). If \( x \in S \), \( v_p(x) \to v_{p+1}(x) \) and \( w_p(x) \to w_{p+1}(x) \).

If both \( \alpha \) and \( \beta \) are in \( C_1(S) \) then both \( \max \{ \alpha, \beta \} \) and \( \min \{ \alpha, \beta \} \) are also in \( C_1(S) \). From this and property (2) of \( \S 2.3 \) it follows that \( v_p \) and \( w_p \) are each in \( C_1(S) \).

By property (1) of \( \S 2.3 \) for each positive integer \( p \), there exist a sequence \( \{ l_{p,n} \}_{n=1}^{\infty} \), \( l_{p,n} \in L(S) \), \( l_{p,n} \geq l_{p,n+1} \), converging to \( v_p \) on \( S \) and a sequence \( \{ u_{p,n} \}_{n=1}^{\infty} \), \( u_{p,n} \in U(S) \), \( u_{p,n} \leq u_{p,n+1} \), converging to \( w_p \) on \( S \). Define \( m_p = \min \{ l_1, p, l_2, p, \ldots, l_p, p \} \) and \( q_p = \max \{ u_1, p, u_2, p, \ldots, u_p, p \} \). The sequence \( \{ m_p \}_{p=1}^{\infty} \), \( m_p \in L(S) \), \( m_p \geq m_{p+1} \), converges to \( f \) on \( \mathbb{R} \). The sequence \( \{ q_p \}_{p=1}^{\infty} \), \( q_p \in U(S) \), \( q_p \leq q_{p+1} \), converges to \( f \) on \( \mathbb{R} \). Further, \( m_p \geq q_p \).

Define \( g \) to be the average of the limit of \( \{ m_p \}_{p=1}^{\infty} \) and the limit of \( \{ q_p \}_{p=1}^{\infty} \). As both sequences converge to \( f \) on \( \mathbb{R} \), \( g \) agrees with \( f \) on \( R \). Define \( \alpha_p = m_{p+1} + 1/p \) and \( \beta_p = q_p + 1/p \). As \( \alpha_p \in L(S) \), \( \beta_p \in U(S) \), and \( \alpha_p > \beta_p \), by a theorem due to Nagami \[3\] there exists a function \( g_p \in C(S) \) such that \( \alpha_p > g_p > \beta_p \). Denote the points of \( T \) as \( \{ x_1, x_2, \ldots \} \). The function \( g \) can be chosen so that it agrees with \( g \) at the first \( p \) points of \( T \). This statement can be justified as follows: Consider the first two points of \( T \), \( x_1 \) and \( x_2 \). If every neighborhood of \( x_1 \) contains \( x_2 \) or vice versa then every continuous function has the same value at \( x_1 \) and \( x_2 \). This is also true of the functions in \( C_1(S) \), in particular those in \( U(S) \) and \( L(S) \). Thus, \( g(x_1) = g(x_2) \). As \( S \) is perfectly normal, \( x_1 \) and \( x_2 \) are contained in a closed subset of a neighborhood \( V \) of \( x_1 \) or \( x_2 \) which has the property that there exist numbers \( a_1 \) and \( a_2 \) such that if \( z \in V \), \( \alpha_p(z) > a_1 > g(x_1) > a_2 > \beta_p(z) \). If there is a neighborhood of \( x_1 \) which does not contain \( x_2 \) and vice versa, then there exist a neighborhood \( V \) of \( x_1 \) and a neighborhood \( W \) of \( x_2 \) such that \( x_1 \) belongs to a closed subset of \( V \) which does not intersect \( W \) and there exist numbers \( a_1 \) and \( a_2 \) such that if \( z \in V \), \( \alpha_p(z) > a_1 > g(x_1) > a_2 > \beta_p(z) \). The neighborhood \( W \) has similar properties. In either case Urysohn’s lemma can be used to modify \( g_p \) such that it agrees with \( g \) at \( x_1 \) and \( x_2 \) but is still continuous and between \( \alpha_p \) and \( \beta_p \). This process can be continued for a finite number of points of \( T \).

The sequence \( \{ g_p \}_{p=1}^{\infty} \) converges to \( g \). If \( x \in R \), \( f(x) = g(x) \).

(c) \( 3 \to 1 \): Suppose \( \{ g_p \}_{p=1}^{\infty}, g_p \in C(S) \), converges to \( g \). Denote the members of \( T \) as \( \{ x_1, x_2, \ldots \} \). For each positive integer \( p \) define
\[ f_p(x) = f(x) \quad \text{if } x = x_n, \quad n \leq p, \]
\[ = g_p(x) \quad \text{if } x \neq x_n, \quad n \leq p, \]

for \( x \in S \). The function \( f_p \) has at most a finite number of discontinuities, each of which is removable. The sequence \( \{f_p\}_{p=1}^{\infty} \) converges to \( f \).

**References**


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