LIMIT OF A SEQUENCE OF FUNCTIONS WITH ONLY COUNTABLY MANY POINTS OF DISCONTINUITY

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1. Introduction and statement of results.

1.1. *Introduction.* We present here first an approximation theorem (Theorem 1) for certain limit functions defined on a general topological space. This strengthens a result which may be found in Hausdorff [1]. With the aid of this theorem we characterize the limits of some classes of discontinuous functions in Theorem 2.

Denote by S a topological space. All functions considered are real valued. Convergence means pointwise convergence unless otherwise stated. Suppose f is a function defined on S, $x \in S$, and f is not continuous at x. The statement that $(x, f(x))$ is a removable point of discontinuity means that there exists a function g which agrees with f on $S - \{x\}$ and which is continuous at x. The statement that the function u defined on S is upper semicontinuous means that, if $x \in S$ and $d > u(x)$, then there exists a neighborhood V of x such that, if $y \in V$, then $d > u(y)$. The function l is lower semicontinuous if $-l$ is upper semicontinuous.

1.2. *Statement of Theorems.*

THEOREM 1. *Suppose M is a linear space of real valued functions defined on S which contains a nonzero constant function and which is closed under the operation of absolute value, and U is the set to which u belongs only in case u is the greatest lower bound of a countable subset of M. Then, if the function f defined on S is the limit of a sequence of functions in M, it is the uniform limit of a sequence each term of which is the difference of two members of U, each of which is bounded above.*

THEOREM 2. *Suppose S is perfectly normal and f is a function defined on S. Each two of the following three statements are equivalent:*

(1) *the function f is the limit of a sequence of functions, each of which has at most a finite number of points of discontinuity, each of which is removable;*

(2) *the function f is the limit of a sequence of functions, each of which has at most countably many points of discontinuity; and*

(3) *there exist a function g which is the limit of a sequence of continuous functions and a countable subset T of S such that, if $x \in S - T$, $f(x) = g(x)$.*

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2. **Proof of theorems.**

2.1. *Proof of Theorem* 1. The following facts should be noted: As M is closed under the operation of absolute value, if each of h and k is in M then each of $\max\{h, k\}$ and $\min\{h, k\}$ is in M. Also, each member of U is the limit from above of a monotonic sequence of members of M.

Suppose $\{f_p\}_{p=1}^{\infty}$, $f_p \in M$, converges to f. For each positive integer p define $l_p = \text{l.u.b.}\{f_p, f_{p+1}, \cdots\}$ and $u_p = \text{g.l.b.}\{f_p, f_{p+1}, \cdots\}$. Thus $-l_p \in U$, $u_p \in U$, $l_p \geqq f_p \geqq u_p$, $l_{p+1} \leqq l_p$, and $u_p \leqq u_{p+1}$. Suppose $c > 0$. Define $R_p = \{x \mid l_p(x) - u_p(x) \leqq c\}$. We assume that R_1 contains at least one point. Note that $\bigcup_{p=1}^{\infty} R_p = S$.

Now we show that the function g_p defined such that $g_p(x) = 1$ if $x \in R_p$ and $g_p(x) = 0$ if $x \notin R_p$ is in U. There exists a monotonic sequence $\{v_n\}_{n=1}^{\infty}$, $v_n \in M$, converging from above to u_p and there exists a monotonic sequence $\{w_n\}_{n=1}^{\infty}$, $w_n \in M$, converging from below to l_p. Define $h_n = w_n - v_n$ and $Q_n = \{x \mid h_n(x) \leqq c\}$. Thus $R_p = \bigcap_{n=1}^{\infty} Q_n$. Define $q_n(x) = 1$ if $x \in Q_n$ and $q_n(x) = 0$ if $x \notin Q_n$. Define $d_n = \max\{1 - (\max\{h_n, c\} - c), 0\}$. Now $d_n(x) = 1$ if $x \in Q_n$ and $0 \leqq d_n(x) < 1$ if $x \notin Q_n$. Define $r_{n,i} = \max\{i \cdot d_n - i + 1, 0\}$. Thus $\{r_{n,i}\}_{i=1}^{\infty}$, $r_{n,i} \in M$, is a monotonic sequence converging to q_n from above. Define $\alpha_{i,p} = \min\{r_{1,i}, r_{2,i}, \cdots, r_{i,i}\}$. Then it is true that $\{\alpha_{i,p}\}_{i=1}^{\infty}$, $\alpha_{i,p} \in M$, is a monotonic sequence converging to g_p from above and therefore $g_p \in U$.

Define $f_0 = 0$,

$$s_p = \sum_{n=1}^{p} \max\{f_n - f_{n-1}, 0\},$$

and

$$t_p = \sum_{n=1}^{p} \min\{f_n - f_{n-1}, 0\}.$$

Note that $s_p + t_p = f_p$. Define $h(x) = s_p(x)$ and $k(x) = t_p(x)$ if $x \in R_p$ but $x \notin R_{p-1}$. Note that both k and $-h$ are bounded above.

Now we show that each of k and $-h$ is in U. Define $\beta_{i,p} = i \cdot \alpha_{i,p} - i$. Thus $\beta_{i,p}(x) = 0$ for $x \in R_p$, $\beta_{i,p}(x) < 0$ for $x \notin R_p$, and $\beta_{i,p}(x) \to -\infty$ as $i \to \infty$ if $x \notin R_p$. Define $\delta_{i,1} = \max\{\beta_{i,1} + t_1, t_2\}$. Define $\delta_{i,n} = \max\{\beta_{i,n} + \delta_{i,n-1}, t_{n+1}\}$ if $n > 1$. Define $\gamma_i = \delta_{i,i}$. If $p \leqq i$ and $x \in R_p$, $\gamma_i(x) = \delta_{i,p}(x)$. Thus $\{\gamma_i\}_{i=1}^{\infty}$, $\gamma_i \in M$, is a monotonic sequence converging from above to k and $k \in U$. A similar argument shows that $-h$ is in U.

If $x \in R_p$ but $x \notin R_{p-1}$, then $h(x) = s_p(x)$, $k(x) = t_p(x)$, and $h(x) + k(x) = f_p(x)$. By the way R_p was defined

$$|f(x) - h(x) - k(x)| = |f(x) - f_p(x)| \leqq c.$$

2.2. *Notation.* Suppose R is a subset of S. Denote by $U(R)$, $L(R)$, and $C(R)$, respectively, the set of all upper semicontinuous functions, lower semicontinuous functions, and continuous functions defined on R. Denote by $C_1(R)$ the set of all functions which are the limit of a sequence of members of $C(R)$. Denote by $U(R)+L(R)$ the set $\{f|f=h+k,\ h\in U(R)k\in L(R)\}$.

2.3. *Some properties of semicontinuous functions.* Certain facts concerning semicontinuous functions should be recalled. Property (1) is that if $\{f_p\}_{p=1}^{\infty}, f_p\in C(S)$, converges to f, then there exists a sequence $\{u_p\}_{p=1}^{\infty}$, $u_p\in U(S)$, $u_{p+1}\geqq u_p$, converging to f and a sequence $\{l_p\}_{p=1}^{\infty}$, $l_p\in L(S)$, $l_{p+1}\leqq l_p$, converging to f such that $l_p\geqq f_p\geqq u_p$. This can be verified by defining $u_p=\text{g.l.b. } \{f_n|n=p,\ p+1,\ \cdots\}$ and $l_p= \text{l.u.b. } \{f_n|n=p,\ p+1,\ \cdots\}$. Property (2) is that S is perfectly normal if and only if every $u\in U(S)$ is the limit from above of a monotonic sequence of continuous functions. This is a theorem due to Hing Tong [2].

2.4 *Proof of Theorem 2.*

(a) $1\rightarrow 2$: This follows immediately.

(b) $2\rightarrow 3$: Suppose $\{f_p\}_{p=1}^{\infty}$ is a sequence of functions defined on S converging to f such that each term of the sequence has at most countably many points of discontinuity. Define $T=\{x|f_p$ is discontinuous at x for some positive integer $p\}$. The set T is countable. Define $R=S-T$, r_p to be the restriction of f_p to R, and r to be the restriction of f to R. Thus, $\{r_p\}_{p=1}^{\infty}$, $r_p\in C(R)$, converges to r. Now we apply Theorem 1. We take M to be the set of all continuous functions defined on R. The limit from above of a monotonic sequence of continuous functions is upper semicontinuous. Therefore, the set U of Theorem 1 is a subset of $U(R)$. Thus, r is the uniform limit of a sequence $\{b_p\}_{p=1}^{\infty}$, where each b_p is the difference of two upper semicontinuous functions, each bounded above. A function $u\in U(R)$ which is bounded above can be extended to a function $v\in U(S)$ by defining $v(x)=\text{g.l.b. } \{\text{l.u.b. } \{u(y)|y\in V\}|\ V$ a neighborhood of $x\}$ where this lower bound exists. There exist at most a countable number of points $x_1,\ x_2,\ \cdots$ of S where the lower bound does not exist. At these points define $v(x_p)=-p$. By this means b_p can be extended to a function $d_p\in U(S)+L(S)$. Since $\{d_p\}_{p=1}^{\infty}$ converges uniformly to f on R, we can assume without loss of generality that $|d_p(x)-f(x)| <1/p$ for $x\in R$ and all p. Define $c_p=d_p+1/p$ and $e_p=d_p-1/p$. If $x\in R$ and p and j are positive integers, $c_p(x)\geqq f(x)\geqq e_j(x)$. Also, if $x\in R$, $c_p(x)\rightarrow f(x)$ as $p\rightarrow\infty$ and $e_p(x)\rightarrow f(x)$ as $p\rightarrow\infty$.

Define $s_p = \min\{c_1, c_2, \cdots, c_p\}$ and $t_p = \max\{e_1, e_2, \cdots, e_p\}$. If $x \in R$, $s_p(x) \geqq f(x) \geqq t_p(x)$. If $x \in S$, $s_p(x) \geqq s_{p+1}(x)$ and $t_p(x) \leqq t_{p+1}(x)$. Define $v_p = \min\{\max\{s_1, t_1\}, \max\{s_2, t_2\}, \cdots, \max\{s_p, t_p\}\}$ and $w_p = \max\{\min\{s_1, t_1\}, \min\{s_2, t_2\}, \cdots, \min\{s_p, t_p\}\}$. If $x \in R$, $v_p(x) \to f(x)$ as $p \to \infty$ and $w_p(x) \to f(x)$ as $p \to \infty$. If $x \in S$, $v_p(x) \geqq v_{p+1}(x) \geqq w_{p+1}(x) \geqq w_p(x)$.

If both α and β are in $C_1(S)$ then both $\max\{\alpha, \beta\}$ and $\min\{\alpha, \beta\}$ are also in $C_1(S)$. From this and property (2) of §2.3 it follows that v_p and w_p are each in $C_1(S)$.

By property (1) of §2.3, for each positive integer p, there exist a sequence $\{l_{p,n}\}_{n=1}^{\infty}$, $l_{p,n} \in L(S)$, $l_{p,n} \geqq l_{p,n+1}$, converging to v_p on S and a sequence $\{u_{p,n}\}_{n=1}^{\infty}$, $u_{p,n} \in U(S)$, $u_{p,n} \leqq u_{p,n+1}$, converging to w_p on S. Define $m_p = \min\{l_{1,p}, l_{2,p}, \cdots, l_{p,p}\}$ and $q_p = \max\{u_{1,p}, u_{2,p}, \cdots, u_{p,p}\}$. The sequence $\{m_p\}_{p=1}^{\infty}$, $m_p \in L(S)$, $m_p \geqq m_{p+1}$, converges to f on R. The sequence $\{q_p\}_{p=1}^{\infty}$, $q_p \in U(S)$, $q_p \leqq q_{p+1}$, converges to f on R. Further, $m_p \geqq q_p$.

Define g to be the average of the limit of $\{m_p\}_{p=1}^{\infty}$ and the limit of $\{q_p\}_{p=1}^{\infty}$. As both sequences converge to f on R, g agrees with f on R. Define $\alpha_p = m_p + 1/p$ and $\beta_p = q_p - 1/p$. As $\alpha_p \in L(S)$, $\beta_p \in U(S)$, and $\alpha_p > \beta_p$, by a theorem due to Nagami [3] there exists a function $g_p \in C(S)$ such that $\alpha_p > g_p > \beta_p$. Denote the points of T as $\{x_1, x_2, \cdots\}$. The function g_p can be chosen so that it agrees with g at the first p points of T. This statement can be justified as follows: Consider the first two points of T, x_1 and x_2. If every neighborhood of x_1 contains x_2 or vice versa then every continuous function has the same value at x_1 and x_2. This is also true of the functions in $C_1(S)$, in particular those in $U(S)$ and $L(S)$. Thus, $g(x_1) = g(x_2)$. As S is perfectly normal, x_1 and x_2 are contained in a closed subset of a neighborhood V of x_1 or x_2 which has the property that there exist numbers a_1 and a_2 such that if $z \in V$, $\alpha_p(z) > a_1 > g(x_1) > a_2 > \beta_p(z)$. If there is a neighborhood of x_1 which does not contain x_2 and vice versa, then there exist a neighborhood V of x_1 and a neighborhood W of x_2 such that x_1 belongs to a closed subset of V which does not intersect W and there exist numbers a_1 and a_2 such that if $z \in V$, $\alpha_p(z) > a_1 > g(x_1) > a_2 > \beta_p(z)$. The neighborhood W has similar properties. In either case Urysohn's lemma can be used to modify g_p such that it agrees with g at x_1 and x_2 but is still continuous and between α_p and β_p. This process can be continued for a finite number of points of T.

The sequence $\{g_p\}_{p=1}^{\infty}$ converges to g. If $x \in R$, $f(x) = g(x)$.

(c) $3 \to 1$: Suppose $\{g_p\}_{p=1}^{\infty}$, $g_p \in C(S)$, converges to g. Denote the members of T as $\{x_1, x_2, \cdots\}$. For each positive integer p define

$$
\begin{aligned}
f_p(x) &= f(x) \quad \text{if } x = x_n, \qquad n \leqq p, \\
&= g_p(x) \quad \text{if } x \neq x_n, \qquad n \leqq p,
\end{aligned}
$$

for $x \in S$. The function f_p has at most a finite number of discontinuities, each of which is removable. The sequence $\{f_p\}_{p=1}^{\infty}$ converges to f.

References

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