HOMOTOPY TYPES OF ARNS'S

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Mather (Topology 4 (1965), 92–93) showed that there were only a countable number of homotopy types of compact topological manifolds. His proof used Whitehead's celebrated theorem on homotopy types. The purpose of this note is to give a short direct elementary point-set proof of the following generalization of Mather's result:

**Theorem.** There are only countably-many homotopy types of compact metric ANR's.

**Proof.** Suppose not. Let \( \{A_\beta\} \) be an uncountable collection of different types. We may regard each \( A_\beta \) as a subset of the Hilbert cube \( I^\omega \). For each \( \beta \) let \( \epsilon_\beta \) be chosen so that \( A_\beta \) is a retract of its \( \epsilon_\beta \)-neighborhood \( U_\beta \) in \( I^\omega \). Let \( r_\beta: U_\beta \rightarrow A_\beta \) be a retraction map. Since \( \{A_\beta\} \) is uncountable we may assume (by choosing a subcollection) that there is a positive \( \epsilon \) so that \( \epsilon_\beta \geq \epsilon \) for each \( \beta \).

Note that if \( f \) and \( g \) are any two maps of a space \( X \) into an \( A_\beta \) so that the distance in \( I^\omega \) between \( f(x) \) and \( g(x) \) is less than \( \epsilon \), for all \( x \) in \( X \), then \( f \) and \( g \) are homotopic. In fact \( F(x, t) = r_\beta((1-t)f(x) + tg(x)) \) is the desired homotopy, where we use the linear structure in \( I^\omega \).

For each \( \beta \) there is a \( \delta_\beta > 0 \) so that \( r_\beta \) restricted to the \( \delta_\beta \)-neighborhood of \( A_\beta \) moves no point as far as \( \epsilon/2 \). Again we may assume that there is a positive \( \delta \) so that \( \delta_\beta \geq \delta \) for each \( \beta \).

The hyperspace of all closed subsets of \( I^\omega \) with the Hausdorff metric is separable (e.g. if \( D \) is any countable dense set in \( I^\omega \), the collection of finite subsets of \( D \) is countable and dense in the hyperspace). Since every uncountable set, in a space satisfying the Second Axiom of Countability, contains a limit point, we see that for some \( \beta \) and \( \gamma \), \( A_\beta \) is in the \( \delta \)-neighborhood of \( A_\gamma \) and vice versa. Then \( r_\gamma | A_\beta \) and \( r_\beta | A_\gamma \) are homotopy inverses of each other, since their compositions in either direction move points a distance less than \( \epsilon \), hence, are homotopic to the corresponding identity maps. This shows \( A_\beta \) and \( A_\gamma \) have the same homotopy type, a contradiction.

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Received by the editors November 29, 1966.