HOMOTOPY TYPES OF ARNS'S

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Mather (Topology 4 (1965), 92–93) showed that there were only a countable number of homotopy types of compact topological manifolds. His proof used Whitehead's celebrated theorem on homotopy types. The purpose of this note is to give a short direct elementary point-set proof of the following generalization of Mather's result:

Theorem. There are only countably-many homotopy types of compact metric ANR's.

Proof. Suppose not. Let \{A_\beta\} be an uncountable collection of different types. We may regard each \(A_\beta\) as a subset of the Hilbert cube \(I^\omega\). For each \(\beta\) let \(\epsilon_\beta\) be chosen so that \(A_\beta\) is a retract of its \(\epsilon_\beta\)-neighborhood \(U_\beta\) in \(I^\omega\). Let \(r_\beta: U_\beta \rightarrow A_\beta\) be a retraction map. Since \{\(A_\beta\)\} is uncountable we may assume (by choosing a subcollection) that there is a positive \(\epsilon\) so that \(\epsilon_\beta \geq \epsilon\) for each \(\beta\).

Note that if \(f\) and \(g\) are any two maps of a space \(X\) into an \(A_\beta\) so that the distance in \(I^\omega\) between \(f(x)\) and \(g(x)\) is less than \(\epsilon\), for all \(x\) in \(X\), then \(f\) and \(g\) are homotopic. In fact \(F(x, t) = r_\beta((1-t)f(x) + tg(x))\) is the desired homotopy, where we use the linear structure in \(I^\omega\). For each \(\beta\) there is a \(\delta_\beta > 0\) so that \(r_\beta\) restricted to the \(\delta_\beta\)-neighborhood of \(A_\beta\) moves no point as far as \(\epsilon/2\). Again we may assume that there is a positive \(\delta\) so that \(\delta_\beta \geq \delta\) for each \(\beta\).

The hyperspace of all closed subsets of \(I^\omega\) with the Hausdorff metric is separable (e.g. if \(D\) is any countable dense set in \(I^\omega\), the collection of finite subsets of \(D\) is countable and dense in the hyperspace). Since every uncountable set, in a space satisfying the Second Axiom of Countability, contains a limit point, we see that for some \(\beta\) and \(\gamma\), \(A_\beta\) is in the \(\delta\)-neighborhood of \(A_\gamma\) and vice versa. Then \(r_\gamma | A_\beta\) and \(r_\beta | A_\gamma\) are homotopy inverses of each other, since their compositions in either direction move points a distance less than \(\epsilon\), hence, are homotopic to the corresponding identity maps. This shows \(A_\beta\) and \(A_\gamma\) have the same homotopy type, a contradiction.

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