ON TOTALLY PARACOMPACT METRIC SPACES[1]

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A topological space is said to be *totally paracompact* [4] if every open basis contains a locally finite cover. It is known [2] that no reflexive infinite-dimensional Banach space is totally paracompact. It is also known [1], [3] that the space of irrationals with usual topology is not totally paracompact. In the present paper we prove a theorem which gives a necessary condition in order that a subset of a complete metric space be totally paracompact. This allows us to exhibit some pathology (Corollary 2) among the separable metric spaces which are not totally paracompact.

Let γ be a collection of sets. We denote by $|\gamma|$ the union of elements of γ. The collection γ is called *point-finite* if no point belongs to infinitely many elements of γ. By a *Cantor set* we mean a set homeomorphic with the Cantor ternary set.

LEMMA. *Let X be a complete metric space and $Y\subset X\backslash Z$ where Z is a nonempty subset of X such that no point of Z is isolated in Z. There exists an open basis β of Y such that if $\gamma\subset\beta$ is point-finite, then $X\backslash|\gamma|$ contains a Cantor set.*

PROOF. We take an increasing infinite sequence $A_1\subset A_2\subset\cdots$ of nonempty countable compact subsets A_n of Z such that no point of A_n is isolated in A_{n+1} for $n=1, 2, \cdots$. Let $U_{n1}\supset U_{n2}\supset\cdots$ be open subsets of X such that

$$A_n = U_{n1}\cap U_{n2}\cap\cdots = \mathrm{Cl}\, U_{n1}\cap \mathrm{Cl}\, U_{n2}\cap\cdots$$

and let $G_n(1), \cdots, G_n(k_n)$ be open subsets of X such that

$$A_n\subset G_n(1)\cup\cdots\cup G_n(k_n), \quad \mathrm{diam}\, G_n(i) < n^{-1}$$

for $i=1, \cdots, k_n$. Let V_n be an open subset of X such that

$$A_n\subset V_n, \quad \mathrm{Cl}\, V_n\subset G_n(1)\cup\cdots\cup G_n(k_n)$$

and let β_n' denote the collection of all open subsets of $Y\backslash \mathrm{Cl}\, V_n$.

We put

$$\beta_n = \beta_n'\cup\{[G_n(i)\backslash \mathrm{Cl}\, U_{nj}]\cap Y \mid i = 1, \cdots, k_n; j = 1, 2, \cdots\}$$

and we define $\beta=\beta_1\cup\beta_2\cup\cdots$. Since $Y\subset X\backslash A_n$, each point from

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[1] This research had partially been done when the author was visiting the Louisiana State University and the University of Wisconsin.

$Y \cap \mathrm{Cl}\, V_n$ belongs to a set $G_n(i) \backslash \mathrm{Cl}\, U_{nj}$ whose diameter is less than n^{-1}. Therefore β is an open basis of Y.

Consider an arbitrary point-finite collection $\gamma \subset \beta$. The collection $\beta_n \backslash \beta_n'$ splits into k_n increasing infinite sequences. Setting $\gamma_n = \beta_n \cap \gamma$ we conclude that $A_n \subset X \backslash \mathrm{Cl} |\gamma_n|$ for $n = 1, 2, \cdots$. No point of A_1 is isolated in A_2, and so there exist two points p_0, p_1 of A_2 belonging to $X \backslash \mathrm{Cl} |\gamma_1|$. We take an open neighborhood $W(m)$ of p_m $(m = 0, 1)$ such that

$$W(m) \subset X \backslash \mathrm{Cl} \mid \gamma_1 \mid, \quad \mathrm{Cl}\, W(0) \cap \mathrm{Cl}\, W(1) = \varnothing,$$

and diam $W(m) < 1$. Let us assume $n > 1$ is an integer, and an open neighborhood $W = W(m_1, \cdots, m_{n-1})$ of a point from A_n is defined, where the indices $m_1, \cdots, m_{n-1}$ are 0 or 1. Then there exist two points q_0, q_1 of A_{n+1} belonging to $W \backslash \mathrm{Cl} |\gamma_n|$. We take an open neighborhood $W(m_1, \cdots, m_{n-1}, m)$ of q_m $(m = 0, 1)$ such that

$$\mathrm{Cl}\, W(m_1, \cdots, m_{n-1}, m) \subset W(m_1, \cdots, m_{n-1}) \backslash \mathrm{Cl} \mid \gamma_n \mid,$$
$$\mathrm{Cl}\, W(m_1, \cdots, m_{n-1}, 0) \cap \mathrm{Cl}\, W(m_1, \cdots, m_{n-1}, 1) = \varnothing,$$

and diam $W(m_1, \cdots, m_{n-1}, m) < n^{-1}$. The intersection

$$C = \bigcap_{n=1}^{\infty} \bigcup_{(m_1, \cdots, m_n)} \mathrm{Cl}\, W(m_1, \cdots, m_n)$$

is a Cantor set, and $C \subset X \backslash (|\gamma_1| \cup |\gamma_2| \cup \cdots) = X \backslash |\gamma|$.

THEOREM. *Let X be a complete metric space and $Y \subset X$. If Y is totally paracompact and $p \in X \backslash Y$, then either $X \backslash Y$ contains a Cantor set containing p or each subset of $X \backslash Y$ containing p has an isolated point.*

PROOF. Suppose there exists a set $Z \subset X \backslash Y$ such that $p \in Z$ and no point of Z is isolated in Z. Applying the lemma to the sets

$$X_n = \{x \mid \mathrm{dist}\,(p, x) \leqq n^{-1}\},$$
$$Y_n = X_n \cap Y, \quad Z_n = \mathrm{Int}\, X_n \cap Z,$$

we get a Cantor set $C_n \subset X_n \backslash Y_n$ because Y_n is totally paracompact $(n = 1, 2, \cdots)$. The union $\{p\} \cup C_1 \cup C_2 \cup \cdots$ is a Cantor set contained in $X \backslash Y$.

COROLLARY 1. *Let X be a complete separable metric space and $Y \subset X$. If Y is uncountable and Y contains no Cantor set, then $X \backslash Y$ is not totally paracompact.*

Each closed subset of a totally paracompact space is totally paracompact. Thus if X is totally paracompact, no closed subset of X is homeomorphic with the irrationals. Answering a question raised by E. Michael we prove the inverse implication is not true. Namely, as a result of Bernstein's construction [**5**, p. 422] which uses the axiom of choice one can obtain a subset X of the Euclidean $(n+1)$-space E^{n+1} such that both X and $E^{n+1}\setminus X$ have the cardinality of the continuum and contain no Cantor sets.

COROLLARY 2. *If the axiom of choice is true, then there exists, for every $n \geqq 0$, an n-dimensional subset X of the Euclidean $(n+1)$-space such that X is not totally paracompact and X contains no Cantor set.*

Observe that an analogous singularity exists among totally paracompact spaces. In fact, as a result of Lusin's construction [**5**, pp. 432–433] using the axiom of choice and the continuum hypothesis one gets an uncountable subset X of the irrationals such that X is totally paracompact and X contains no Cantor set.

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