A topological space is said to be totally paracompact \cite{4} if every open basis contains a locally finite cover. It is known \cite{2} that no reflexive infinite-dimensional Banach space is totally paracompact. It is also known \cite{1}, \cite{3} that the space of irrationals with usual topology is not totally paracompact. In the present paper we prove a theorem which gives a necessary condition in order that a subset of a complete metric space be totally paracompact. This allows us to exhibit some pathology (Corollary 2) among the separable metric spaces which are not totally paracompact.

Let $\gamma$ be a collection of sets. We denote by $|\gamma|$ the union of elements of $\gamma$. The collection $\gamma$ is called point-finite if no point belongs to infinitely many elements of $\gamma$. By a Cantor set we mean a set homeomorphic with the Cantor ternary set.

**Lemma.** Let $X$ be a complete metric space and $Y \subset X \setminus Z$ where $Z$ is a nonempty subset of $X$ such that no point of $Z$ is isolated in $Z$. There exists an open basis $\beta$ of $Y$ such that if $\gamma \subset \beta$ is point-finite, then $X \setminus |\gamma|$ contains a Cantor set.

**Proof.** We take an increasing infinite sequence $A_1 \subset A_2 \subset \cdots$ of nonempty countable compact subsets $A_n$ of $Z$ such that no point of $A_n$ is isolated in $A_{n+1}$ for $n = 1, 2, \cdots$. Let $U_1 \supset U_2 \supset \cdots$ be open subsets of $X$ such that

$$A_n = U_1 \cap U_2 \cap \cdots = \text{Cl } U_1 \cap \text{Cl } U_2 \cap \cdots$$

and let $G_n(1), \cdots, G_n(k_n)$ be open subsets of $X$ such that

$$A_n \subset G_n(1) \cup \cdots \cup G_n(k_n), \quad \text{diam } G_n(i) < n^{-1}$$

for $i = 1, \cdots, k_n$. Let $V_n$ be an open subset of $X$ such that

$$A_n \subset V_n, \quad \text{Cl } V_n \subset G_n(1) \cup \cdots \cup G_n(k_n)$$

and let $\beta_n'$ denote the collection of all open subsets of $X \setminus \text{Cl } V_n$.

We put

$$\beta_n = \beta_n' \cup \{ [G_n(i) \setminus \text{Cl } U_n] \cap Y \mid i = 1, \cdots, k_n; j = 1, 2, \cdots \}$$

and we define $\beta = \beta_1 \cup \beta_2 \cup \cdots$. Since $Y \subset X \setminus A_n$, each point from

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1 This research had partially been done when the author was visiting the Louisiana State University and the University of Wisconsin.
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Y \cap \text{Cl} V_n belongs to a set \( G_n(i) \setminus \text{Cl} U_{nj} \) whose diameter is less than \( n^{-1} \). Therefore \( \beta \) is an open basis of \( Y \).

Consider an arbitrary point-finite collection \( \gamma \subset \beta \). The collection \( \beta_n \setminus \beta_n' \) splits into \( k_n \) increasing infinite sequences. Setting \( \gamma_n = \beta_n \cap \gamma \) we conclude that \( A_n \cap X \setminus \text{Cl} \left| \gamma_n \right| \) for \( n = 1, 2, \ldots \). No point of \( A_1 \) is isolated in \( A_2 \), and so there exist two points \( p_0, p_1 \) of \( A_2 \) belonging to \( X \setminus \text{Cl} \left| \gamma_1 \right| \). We take an open neighborhood \( W(m) \) of \( p_m \) (\( m = 0, 1 \)) such that

\[
W(m) \subset X \setminus \text{Cl} \left| \gamma_1 \right| , \quad \text{Cl} W(0) \cap \text{Cl} W(1) = \emptyset ,
\]

and \( \text{diam} \ W(m) < 1 \). Let us assume \( n > 1 \) is an integer, and an open neighborhood \( W = W(m_1, \ldots, m_{n-1}) \) of a point from \( A_n \) is defined, where the indices \( m_1, \ldots, m_{n-1} \) are 0 or 1. Then there exist two points \( q_0, q_1 \) of \( A_{n+1} \) belonging to \( W \setminus \text{Cl} \left| \gamma_n \right| \). We take an open neighborhood \( W(m_1, \ldots, m_{n-1}, m) \) of \( q_m \) (\( m = 0, 1 \)) such that

\[
\text{Cl} W(m_1, \ldots, m_{n-1}, m) \subset W(m_1, \ldots, m_{n-1}) \setminus \text{Cl} \left| \gamma_n \right| ,
\]

\[
\text{Cl} W(m_1, \ldots, m_{n-1}, 0) \cap \text{Cl} W(m_1, \ldots, m_{n-1}, 1) = \emptyset ,
\]

and \( \text{diam} \ W(m_1, \ldots, m_{n-1}, m) < n^{-1} \). The intersection

\[
C = \bigcap_{n=1}^{\infty} \bigcup_{(m_1, \ldots, m_n)} \text{Cl} W(m_1, \ldots, m_n)
\]

is a Cantor set, and \( C \subset X \setminus (\left| \gamma_1 \right| \cup \left| \gamma_2 \right| \cup \cdots) = X \setminus \gamma \).

THEOREM. Let \( X \) be a complete metric space and \( Y \subset X \). If \( Y \) is totally paracompact and \( p \in X \setminus Y \), then either \( X \setminus Y \) contains a Cantor set containing \( p \) or each subset of \( X \setminus Y \) containing \( p \) has an isolated point.

PROOF. Suppose there exists a set \( Z \subset X \setminus Y \) such that \( p \in Z \) and no point of \( Z \) is isolated in \( Z \). Applying the lemma to the sets

\[
X_n = \{ x \mid \text{dist} (p, x) \leq n^{-1} \},
\]

\[
Y_n = X_n \cap Y , \quad Z_n = \text{Int} X_n \cap Z ,
\]

we get a Cantor set \( C_n \subset X_n \setminus Y_n \) because \( Y_n \) is totally paracompact \( (n = 1, 2, \ldots) \). The union \( \{ p \} \cup C_1 \cup C_2 \cup \cdots \) is a Cantor set contained in \( X \setminus Y \).

COROLLARY 1. Let \( X \) be a complete separable metric space and \( Y \subset X \). If \( Y \) is uncountable and \( Y \) contains no Cantor set, then \( X \setminus Y \) is not totally paracompact.

Each closed subset of a totally paracompact space is totally paracompact. Thus if \( X \) is totally paracompact, no closed subset of \( X \) is
homeomorphic with the irrationals. Answering a question raised by E. Michael we prove the inverse implication is not true. Namely, as a result of Bernstein's construction [5, p. 422] which uses the axiom of choice one can obtain a subset $X$ of the Euclidean $(n+1)$-space $E^{n+1}$ such that both $X$ and $E^{n+1}\setminus X$ have the cardinality of the continuum and contain no Cantor sets.

**Corollary 2.** If the axiom of choice is true, then there exists, for every $n \geq 0$, an $n$-dimensional subset $X$ of the Euclidean $(n+1)$-space such that $X$ is not totally paracompact and $X$ contains no Cantor set.

Observe that an analogous singularity exists among totally paracompact spaces. In fact, as a result of Lusin's construction [5, pp. 432–433] using the axiom of choice and the continuum hypothesis one gets an uncountable subset $X$ of the irrationals such that $X$ is totally paracompact and $X$ contains no Cantor set.

**Bibliography**


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