A SCHROEDER-BERNSTEIN THEOREM FOR PROJECTIONS[1]

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The purpose of this note is to give the algebraic part of a proof for a result from the theory of von Neumann algebras.

Consider a ring with involution (*) satisfying $(a+b)^* = a^* + b^*$ and $(ab)^* = b^*a^*$. An idempotent e is called a projection if it is selfadjoint ($e = e^*$). The set P of all projections is partially ordered by the relation $\leqq$, defined by setting $e \leqq f$ whenever $ef = e$. A pair of projections, e and f, are said to be equivalent ($e \sim f$) if there is an element v such that $vv^* = e$ and $v^*v = f$. A new relation $\prec$ is defined by $e \prec f$ if $e \sim g \leqq f$.

THEOREM. *If the partially ordered set* $(P, \leqq)$ *is a complete lattice, then* $e \prec f$ *and* $f \prec e$ *imply* $e \sim f$.

In order to make this paper self contained the following well-known results are included.

LEMMA 1. *If* $e_i \sim f_i$ $(i = 1, 2)$, $e_1e_2 = 0$, *and* $f_1f_2 = 0$ *then* $e_1 + e_2 \sim f_1 + f_2$.

PROOF. By assumption there exist v_i such that $v_i^*v_i = f_i$ and $v_iv_i^* = e_i$. Let $w = v_1v_1^*v_1 + v_2v_2^*v_2$, then $ww^* = e_1 + e_2$ and $w^*w = f_1 + f_2$.

LEMMA 2. *An order preserving map on a complete lattice has a fixed point.*

PROOF. Let ϕ be the map and $p = \sup E$ where

$$E = \{g \mid g \leqq \phi(g)\}.$$

For g in E, $g \leqq p$ so that $g \leqq \phi(g) \leqq \phi(p)$, therefore $p \leqq \phi(p)$. It follows that $\phi(p) \leqq \phi(\phi(p))$, implying $\phi(p)$ is in E, hence $\phi(p) \leqq p$.

PROOF OF THE THEOREM. By assumption there are elements v and w such that $w^*w \leqq f$, $ww^* = e$, $v^*v \leqq e$ and $vv^* = f$. For $g \leqq f$ define

$$\phi(g) = f - w^*(e - v^*gv)w.$$

Direct computation shows that $h \leqq g \leqq f$ implies $v^*hv \leqq v^*gv \leqq e$ and $f - g \leqq f - h$. Now it can be seen that ϕ is the composition of four maps of which two are order preserving and two are order reversing. Thus it follows that ϕ is order preserving on the complete lattice $[0, f]$, so that by Lemma 2 there is a projection $p \leqq f$ such that

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$$p = f - w^*(e - v^*pv)w.$$

The elements pv and $(e-v^*pv)w$ implement the equivalences

$$p \sim v^*pv \quad \text{and} \quad f - p \sim e - v^*pv.$$

Now it follows from Lemma 1 that $f \sim e$.

Von Neumann algebras, W^*-algebras, AW^*-algebras and Baer rings are examples of rings with a complete lattice of projections.

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ON THE FOURIER INVERSION THEOREM FOR R^1

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The following is an elementary "noncomputational" proof of the Fourier inversion theorem for tempered distributions on R^1. The proof does not generalize so easily to R^n, but the inversion theorem for R^n can be deduced from that for R^1.

To get the inversion theorem for tempered distributions it is sufficient, by duality, to have a proof for the space $\mathfrak{D}$ of *test functions* (i.e. functions $\phi \in C^\infty$ such that $\phi^{(m)}(x) = O(|x|^{-N})$ for all m, $N \geqq 0$ as $x \to \pm\infty$). It is also sufficient to consider only the point $x = 0$.

THEOREM. *There exists a universal constant K such that*

(1) $\int_{-\infty}^{\infty} \hat{\phi}(t)dt = K\phi(0)$ *for all* $\phi \in \mathfrak{D}$.

The value of K ($K = 2\pi$) must be determined, as usual, by substituting some particular function ϕ. By linearity, (1) *is equivalent to the following:*

(2) $\phi(0) = 0$ *implies* $\int_{-\infty}^{\infty} \hat{\phi}(t)dt = 0$.

PROOF OF (2). Since $\phi(0) = 0$, $\psi(x) \equiv (\phi(x)/x) \in C^\infty$. By direct computation, since $\phi(x) = x\psi(x)$, $\hat{\phi}(t) = i(d/dt)\hat{\psi}(t)$. Then, since $\hat{\psi}$ is also a test function (direct verification), $\int_{-\infty}^{\infty} \hat{\phi}(t)dt = i[\hat{\psi}(\infty) - \hat{\psi}(-\infty)] = 0$. Q.E.D.

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