INVERSE LIMITS OF PERFECTLY NORMAL SPACES

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It is the purpose of this note to establish that the countable inverse limit of perfectly normal topological spaces is again a perfectly normal space and to establish some of the consequences of that fact as it pertains to Moore spaces. In particular, this result yields a new proof of the fact that, if there is a normal nonmetrizable (complete) Moore space, then there is one which is not locally metrizable at any point [2], [6].

Throughout this note, $\mathrm{Cl}(M)$ denotes the closure of the point set M. If $\{X_n, \pi_n^m\}$ is an inverse mapping system with inverse limit X_∞ and n is a positive integer, π_n denotes the projection of X_∞ into X_n (all inverse mapping systems are taken over the set of positive integers directed by $<$).

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THEOREM 1. *If X_∞ is the inverse limit of the inverse mapping system $\{X_n, \pi_n^m\}$ where, for each n, X_n is a topological space in which each closed set is an inner limiting ($=G_\delta$) set and π_n^{n+1} is a mapping of X_{n+1} into X_n, then each closed subset of X_∞ is an inner limiting set.*

PROOF. Suppose that $\{X_n, \pi_n^m\}$ and X_∞ are as in the hypothesis and M is a closed subset of X_∞. There exist sequences $0_{11}, 0_{12}, \cdots; 0_{21}, 0_{22}, \cdots; \cdots$ such that, for each integer n, $0_{n1}, 0_{n2}, \cdots$ is a sequence of domains whose common part is $\mathrm{Cl}(\pi_n(M))$ and, for each positive integer m, 0_{nm} contains $0_{n,m+1}$ and $(\pi_n^{n+1})^{-1}(0_{nm})$ contains $0_{n+1,m}$. Clearly, for each n, $\pi_n^{-1}(0_{nn})$ contains M. Suppose that p is a point of X_∞ not in M such that, for each n, $\pi_n^{-1}(0_{nn})$ contains p. For some positive integer n, $\pi_n(p)$ is not in $\mathrm{Cl}(\pi_n(M))$ and, hence, for some $m>n$, $\pi_n(p)$ is not in 0_{nm}. Then $\pi_m(p)$ is not in 0_{mm}, which is a subset of $(\pi_n^m)^{-1}(0_{nm})$. Thus p is not in $\pi_m^{-1}(0_{mm})$ and M is an inner limiting set.

THEOREM 2. *If X_∞ is the inverse limit of the inverse mapping system $\{X_n, \pi_n^m\}$ where, for each n, X_n is a perfectly normal space and π_n^{n+1} is a mapping of X_{n+1} into X_n, then X_∞ is perfectly normal.*

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PROOF. Let $\{X_n, \pi_n^m\}$ and X_∞ be as in the hypothesis and let H and K be mutually exclusive closed point sets in X_∞. It follows from Theorem 1 that every closed set in X_∞ is an inner limiting set.

There exist sequences $0_{11}, 0_{12}, \cdots; 0_{21}, 0_{22}, \cdots; \cdots$ such that, for each n, $0_{n1}, 0_{n2}, \cdots$ is a sequence of domains in X_n whose common part is $\mathrm{Cl}(\pi_n(H))\cdot\mathrm{Cl}(\pi_n(K))$ and, for each m, 0_{nm} contains $\mathrm{Cl}(0_{n,m+1})$ and $(\pi_n^{n+1})^{-1}(0_{nm})$ contains $\mathrm{Cl}(0_{n+1,m})$. Now, $\mathrm{Cl}(\pi_1(H))\backslash 0_{11}$ and $\mathrm{Cl}(\pi_1(K))\backslash 0_{11}$ are two mutually exclusive closed point sets in X_1, the former does not intersect $\mathrm{Cl}(\pi_1(K))$, and the latter does not intersect $\mathrm{Cl}(\pi_1(H))$. Therefore, there exist in X_1 domains D_1 and E_1 containing $\mathrm{Cl}(\pi_1(H))\backslash 0_{11}$ and $\mathrm{Cl}(\pi_1(K))\backslash 0_{11}$ respectively such that $\mathrm{Cl}(D_1)$ does not intersect $\mathrm{Cl}(E_1)$ or $\mathrm{Cl}(\pi_1(K))$ and $\mathrm{Cl}(E_1)$ does not intersect $\mathrm{Cl}(\pi_1(H))$. Similarly, there exist in X_2 domains D_2 and E_2 containing $(\pi_1^2)^{-1}(\mathrm{Cl}(D_1))+[\mathrm{Cl}(\pi_2(H))\backslash 0_{22}]$ and $(\pi_1^2)^{-1}(\mathrm{Cl}(E_1))+[\mathrm{Cl}(\pi_2(K))\backslash 0_{22}]$ respectively such that $\mathrm{Cl}(D_2)$ does not intersect $\mathrm{Cl}(E_2)$ or $(\pi_1^2)^{-1}(\mathrm{Cl}(E_1))+\mathrm{Cl}(\pi_2(K))$ and $\mathrm{Cl}(E_2)$ does not intersect $(\pi_1^2)^{-1}(\mathrm{Cl}(E_1))+\mathrm{Cl}(\pi_2(H))$. This process may be continued to obtain sequences $D_1, D_2, D_3, \cdots$ and $E_1, E_2, E_3, \cdots$ such that, for each positive integer n,

(i) D_n and E_n are mutually exclusive domains in X_n;

(ii) $\pi_n^{-1}(D_n)$ is a subset of $\pi_{n+1}^{-1}(D_{n+1})$ and $\pi_n^{-1}(E_n)$ is a subset of $\pi_{n+1}^{-1}(E_{n+1})$; and

(iii) D_n contains $\mathrm{Cl}(\pi_n(H))\backslash 0_{nn}$ and E_n contains $\mathrm{Cl}(\pi_n(K))\backslash 0_{nn}$.

Let $D=\pi_1^{-1}(D_1)+\pi_2^{-1}(D_2)+\cdots$ and $E=\pi_1^{-1}(E_1)+\pi_2^{-1}(E_2)+\cdots$.

It follows from (i) and (ii) that D and E are mutually exclusive domains in X_∞. It remains to be proved that D contains H and E contains K.

Suppose that p is a point of H. Then p is not in K and there exists a positive integer n such that $\pi_n(p)$ is not in $\mathrm{Cl}(\pi_n(K))$. There is a positive integer $m>n$ such that $\pi_n(p)$ is not in 0_{nm}. Now, $\pi_m(p)$ is not in $(\pi_n^m)^{-1}(0_{nm})$ and is, therefore, not in 0_{mm}. Thus $\pi_m(p)$ is in D_m and, hence, p is in $\pi_m^{-1}(D_m)$. Similarly, E contains K.

The following Theorem of Katetov [**3**] is a corollary of Theorem 2.

THEOREM OF KATETOV. *If all spaces* $P_1\times\cdots\times P_n$ $(n=1, 2, \cdots)$ *are perfectly normal, then the space* $P=P_1\times P_2\times\cdots$ *is perfectly normal as well.*

PROOF. For each n, let π_n^{n+1} be the projection of $P_1\times\cdots\times P_{n+1}$ onto $P_1\times\cdots\times P_n$. Then the inverse limit of the inverse mapping system so obtained is topologically equivalent to P and is perfectly normal.

REMARK. Katetov has shown, [**3**], by an example that his above

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stated theorem does not hold if perfectly normal is replaced by completely normal. Therefore, our Theorem 2 does not hold if perfectly normal is replaced by completely normal even if all of the bonding mappings are onto.

THEOREM 3. *If* X_∞ *is the inverse limit of the inverse mapping system* $\{X_n, \pi_n^m\}$ *where, for each* n, X_n *is a (complete) normal Moore space and* π_n^{n+1} *is a mapping of* X_{n+1} *into* X_n, *then* X_∞ *is a (complete) normal Moore space.*

PROOF. It is known (see, for example, [1] and [7]) that the Cartesian product of countably many (complete) Moore spaces is a (complete) Moore space. Every subspace of a Moore space is a Moore space and every closed subspace of a complete Moore space is complete, [5]. Thus, the countable inverse limit of Moore spaces is a Moore space and, since the inverse limit is closed in the Cartesian product space [4, p. 31], it is complete if the coordinate spaces are complete. Now, each normal Moore space is perfectly normal and the Theorem is proved.

COROLLARY 1. *If each Cartesian product of two normal Moore spaces is normal, then so is each Cartesian product of countably infinitely many.*

The proof of this Corollary is precisely the same as that for the above Theorem of Katetov, or, as has been pointed out by Bruce Anderson, [1], it may be noted that this corollary follows from the Theorem of Katetov.

D. R. Traylor and the second author of this paper showed [2] that, if S^0 is a Moore space, there is a Moore space S^w such that every open set in S^w contains a copy of S^0 and S^w is normal if S^0 is. Traylor showed [6] that S^w is completable in such a way that the completion of S^w is normal if S^0 is complete and normal. The proofs of these results are simplified considerably by using Theorem 3 in the following:

COROLLARY 2. *If* S^0 *is a Moore space then there is a Moore space* S_∞ *in which every open set contains a topological copy of* S^0 *and* S_∞ *is complete or normal, respectively, if* S^0 *is complete or normal.*

PROOF. Let S^0 be a Moore space. The notation of [2] will be used. Let M^j, $S^j_{p,k}$, and S^j be as in [2], i.e., (1) M^j is a dense subset of S^j, (2) $S^{j+1}_{p,1}, S^{j+1}_{p,2}, \cdots$ is a null sequence, of copies of S^0, which converges in S^{j+1} to the point P of M^j, and (3) $S^{j+1} = S^j + \sum_{P\in M^j} \sum_{k=1}^{\infty} S^{j+1}_{p,k}$. Define π_j^{j+1} by $\pi_j^{j+1}(x) = x$ if x is in S^j and $\pi_j^{j+1}(x) = P$ if there is a positive integer k such that x is in $S^{j+1}_{p,k}$. Then π_j^{j+1} is a continuous

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transformation of S^{j+1} onto S^j. The inverse limit, S_∞, of the inverse mapping system $\{S^n, \pi_n^m\}$ is the desired space.

References

1. Bruce A. Anderson, *Metric topologies*, Ph.D. dissertation, Univ. of Iowa, Iowa City, 1966.

2. B. Fitzpatrick, Jr. and D. R. Traylor, *Two theorems on metrizability of Moore spaces*, Pacific J. Math. **19** (1966), 259–264.

2. M. Katetov, *Complete normality of Cartesian products*, Fund. Math. **35** (1948), 271–274.

4. Solomon Lefschetz, *Algebraic topology*, Amer. Math Soc. Colloq. Publ., Vol. 27, Amer. Math Soc., Providence, R.I., 1942.

5. R. L. Moore, *Foundations of point set theory*, rev. ed., Amer. Math. Soc. Colloq. Publ., Vol. 13, Amer. Math. Soc., Providence, R.I., 1962.

6. D. R. Traylor, *Metrizability and completeness in normal Moore spaces*, Pacific J. Math. **17** (1966), 381–390.

7. J. F. Williams, *An investigation of properties of Cartesian product spaces*, Master's thesis, Auburn Univ., Auburn, Ala., 1965.

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