ON BOUNDED DOMAINS

K. SRINIVASACHARYULU

1. **Introduction.** A classical theorem of Riemann asserts that a simply connected Riemann surface is isomorphic to one of the following surfaces: (i) the complex plane C^2; (ii) the unit circle or (iii) the Riemann sphere. A higher dimensional analogue to Kähler manifolds is the following:

THEOREM. *Let V be a simply connected complete Kähler n-manifold with constant holomorphic curvature k; then V is isomorphic to one of the following*:

(i) *the complex number space* C^n $(k=0)$, *or*
(ii) *the complex unit hypersphere K in C^n* $(k<0)$ *or*
(iii) *the complex projective space* $(k>0)$.[1]

This result can be proved by using a well-known theorem of Bochner on "normal coordinates" [**1**] as in [**5**] or [**6**] or can be proved by using É. Cartan's "repère mobile" as in [**3**].[2] We call these three spaces *normal domains.* On the other hand, it is easy to construct counter examples if any one of the conditions "simply connected" or "complete" is removed from the theorem.

It is well known that the normal domains N listed in the theorem admit a transitive Lie group G of analytic isometries of n^2+2n parameters; thus N is a homogeneous space with the structure group G and with the unitary group $U(n)$ as its isotropy group. Conversely we have the following (cf. [**7**])

THEOREM 1. *Let V be a simply connected,*[3] *complete Kähler manifold of dimension n admitting a group of isometries of dimension n^2+2n; then V is isomorphic to a normal domain.*

PROOF. It is known ([**7**, Lemma a, p. 168]) that the group of isometries $I_0(V)$ acts transitively on V; the argument of ([**7**, p. 169, (b)]) proves that V is of constant holomorphic curvature. Conse-

Received by the editors November 16, 1966.

[1] A complete Kähler manifold with $k>0$ is necessarily compact and is simply connected (cf. [**2**], p. 528).

[2] Igusa and Hawley assume that V is compact but Cartan's method works for complete Kähler manifolds; a proof along these lines has been given by the author in a mimeographed seminar "Séminaire C. Ehresmann" 1961 (Paris) on "Espaces symétriques de É. Cartan."

[3] Cf. footnote 1.

quently, V is isomorphic to a normal domain by the theorem cited above.

2. In what follows, we study the geometry of some bounded domains. Let D be a bounded domain in C^n whose Bergmann metric is *complete* and let us consider the holomorphic curvature (of the Bergmann metric) of D. We first assert that the holomorphic curvature of *D cannot vanish everywhere*; otherwise, D, being complete, there exists an analytic local isometry ϕ of D into C^n by Lemma 3 [**6**]; then ϕ^{-1} can be extended to an analytic mapping of C^n into D which is locally single-valued and isometric (cf. [**6**, Lemma 2]). This is impossible in view of the fact that D is bounded. On the other hand, if the holomorphic curvature of D is strictly positive, then D will be compact (cf. [**2**, p. 527]); consequently, being a compact submanifold of C^n, D has to be a single point. Thus we have the following

PROPOSITION. *Let D be a bounded domain in C^n whose Bergmann metric is complete; then its holomorphic curvature cannot be everywhere positive or everywhere zero.*

Examples of such domains are given by homogeneous bounded domains in C^n. We shall prove below a partial converse, namely the following

THEOREM 2. *Let D be a bounded domain in C^n whose Bergmann metric is complete; if the holomorphic curvature of D is a constant, then D is analytically isometric to a hypersphere K in C^n.*

PROOF. We may assume that D is of constant negative holomorphic curvature by the above Proposition. We can construct, as in Lemma 3 of [**6**], an analytic mapping (possibly multivalued) ϕ of D into K which maps a neighbourhood of $p \in D$ isometrically into a neighbourhood of zero in K. On the other hand, since K is simply connected, there exists a single-valued analytic isometry ψ of K into D by Lemma 3 of [**6**] which extends ϕ^{-1}. We may remark that, if $K(z, \bar{z})$ is the Bergmann kernel of D, then the set $U = \{z \mid K(z, \bar{\xi}) = 0, z \in D, \xi$ a fixed point in $D\}$, is an analytic set of dimension $n-1$ and $\phi: D - U \rightarrow K$ is single valued [**8**] and analytic; moreover $D - U$ is connected. Thus if $(\phi_i(z))$[4] are local-coordinates of ϕ, then each $\phi_i(z)$ is bounded; thus ϕ may be extended to holomorphic mapping of D into K by Hartog's theorem (for $n \geqq 2$). We prove now that $\phi\psi$ is identity on K; clearly $\phi\psi$ is an analytic local isometry of K into K which is identity on a neighbourhood of zero; consequently $\phi\psi$ is identity on K. Similarly,

[4] Recall that $\phi_\alpha = T^{\overline{\beta}\alpha}(\xi, \bar{\xi})(\partial/\partial\bar{\xi}^\beta) \log (K(z, \bar{\xi})/K(\xi, \bar{\xi}))$, $\alpha = 1, \cdots, n$.

we may prove that $\psi\phi$ is identity on D; in fact, since $\psi\phi$ leaves $p\in D$ fixed, it is linear by a well-known theorem of H. Cartan [**4**]. Moreover, since the Jacobian of $\psi\phi$ at the origin is a unitary matrix, it follows that $\psi\phi$ is identity on D.

COROLLARY 1. *Let D be a bounded domain in C^n whose Bergmann metric is complete; if D has constant negative curvature, then D is homogeneous.*

COROLLARY 2. *Let D be as in Corollary* 1; *then D is simply connected.*

It may be pointed out that it is *unknown* whether a bounded homogeneous domain is simply connected or not. In conclusion, we may remark that nothing is known about Kähler manifolds (compact or not) of *strictly negative holomorphic curvature*; in the compact case, these manifolds are *known* to be projective algebraic and admit no one parameter groups of holomorphic transformations.

REFERENCES

1. S. Bochner, *Curvature in Hermitian metric*, Bull. Amer. Math. Soc. **53** (1947), 179–195.

2. R. L. Bishop and S. I. Goldberg, *Some implications of the generalized Gauss-Bonnet theorem*, Trans. Amer. Math. Soc. **112** (1964), 508–535.

3. E. Cartan, *Leçons sur la géométrie des espaces de Riemann*, Gauthier-Villars, Paris, 1946.

4. H. Cartan, *Sur les groupes de transformations analytiques*, Actualités Sci. Ind. 198, Hermann, Paris, 1935.

5. N. S. Hawley, *Constant holomorphic curvature*. Canad. J. Math. **5** (1953), 53–56.

6. J. Igusa, *On the structure of a certain class of Kähler varieties*, Amer. J. Math. **76** (1954), 669–778.

7. A. Lichnerowicz, *Géométrie des groupes de transformations*, Dunod, Paris, 1958.

8. W. T. Martin and S. Bochner, *Several complex variables*, Princeton Univ. Press, Princeton, N. J., 1948.

UNIVERSITÉ DE MONTRÉAL