1. A symmetric kernel $G$ is said to symmetrize the kernel $K$ by composition on the left in case the product $GK$ is symmetric — i.e. in case $GK = KTG$. It follows at once that, if $G$ is a left symmetrizer of $K$, so are $GK$, $GK^2$, $GK^3$, etc., and that the linear manifold spanned by these kernels consists entirely of left symmetrizers of $K$. It also follows that $G^{-1}$, when it exists, satisfies $KG^{-1} = G^{-1}KT$, so that the inverse of $G$ is to be sought among the right symmetrizers of $K$.

The integral equations of first kind of classical potential theory, namely

\[ f(p) = \int_S G(pq) \mu(q) dS_q, \]

\[ g(p) = \int_S D(pq) \nu(q) dS_q \]

arise when the solution of the Dirichlet problem with respect to a surface $S$ is sought in the form of the potential $V[\mu]$ of a surface distribution on $S$ of density $\mu$, and the solution of the Neumann problem is sought in the form of the potential $W[\nu]$ of a double layer on $S$ of moment $\nu$. In this notation, $G(p, q) = 1/(2\pi rpq)$ is the potential at $p(q)$ of a unit mass at $q(p)$, and is symmetric, while $D(p, q) = (\partial^2/\partial n_p \partial n_q)G(pq)$ represents the normal component of force at $p(q)$ due to a unit normal dipole at $q(p)$ and is likewise symmetric. The given boundary values relevant to the (interior or exterior) Dirichlet and Neumann problems are $f(p)$ and $g(p)$, respectively.

In this paper, the concepts of the first paragraph above are applied to the solution of the equations (1) and (2) in the case of a closed, bounded surface $S$ of class $B$ [11, p. 186]. For, it is known [19, §4, p. 344] that $G$ is a left symmetrizer of the kernel

\[ K(p, q) = (\partial/\partial n_p)G(p, q) = \cos(r, n_p)/(2\pi rpq)^2 \]

of the Fredholm-Poincaré integral equations. It will be shown that

\[ D(p, q) = [\cos(n_p, n_q) + 3 \cos(n_p, r) \cos(n_q, r)]/(2\pi rpq)^3 \]
is a right symmetrizer of $K$ and that $DG = K^2 - I$ so that, in effect

$$G^{-1} = -[D + K^2D + K^4D + \cdots], \quad D^{-1} = -[G + GK^2 + GK^4 + \cdots],$$

and the integral equations (1) and (2) are equivalent, respectively, to

(3) $K^2u - u = Df,$

(4) $(K^T)^2v - v = Gg.$

Analogous results may be obtained in the plane.

The problem represented by the integral equation (1) has been discussed by Liapounoff [10] who showed that, when $f$ is such that $W[f]$ admits a regular normal derivative on $S$, it may be written $f = G\mu$. The density $\mu$ is obtained as the difference of the solutions of two integral equations, formulated with respect to the regions interior and exterior to $S$, respectively. For general $f$ he showed that the third and succeeding terms of the Neumann series solution of the Dirichlet problem with boundary values $f$ could be written as the potential of a single layer but, in general, the first two terms could not. Similar results were obtained by E. R. Neumann [14] who, in addition, obtained the solution of the Neumann problem in the form of a double layer potential [14, pp. 43–66]—i.e. solved the integral equation (2). Bertrand [1] converted the equation (1), in the two-dimensional case, to an equation of second kind by differentiation while Plume [21] has given a similar treatment of the Neumann problem. Picard [17], [18], in a well-known paper, has given necessary and sufficient conditions, applicable in the two-dimensional case, that (1) admit a square-integrable solution, and has worked out the case of a circle. These methods have been extended to the three-dimensional case by Fenyo [3], who illustrates his results in the case of a sphere. Blumer [2] converts (1), in the three-dimensional case, to each of three integral equations of second kind by a complicated process based upon integro-differential operators analogous to those of M. Riesz. Thus it appears that the symmetrizing property of $D$ and the equivalence of (1) and (2) with equations of second kind with kernels $K^2$ and $(K^T)^2$ are new results. The following development, however, owes much to the work of Liapounoff [8], [9], [10].

2. A function $V(P)$ harmonic in the region $R$ interior, or $R'$ exterior, to $S$ is said to possess a regular normal derivative on $S$ [9, §2, p. 246, §16, p. 285] in case $\lim_{P \rightarrow P' \in S}(\partial V/\partial n)(P)$ is taken uniformly on $S$ as $P \rightarrow P'$ along the normal to $S$ at $P$. (The normal $n$ is defined throughout as the interior normal to $S$, and determines the positive (interior) and negative (exterior) sides of $S$. See, e.g. equations (6)
and (8).) These limiting values then define a continuous function on $S$. Tauber [22], [23] (see also Liapounoff [8, p. 131]) has shown that the difference of the derivatives of a double layer potential $W[\nu]$, with continuous moment $\nu$ on $S$, in the normal direction at points on the normal on opposite sides and equidistant from $S$, vanishes as these points approach $S$, and hence that, if $W[\nu]$ admits a regular normal derivative on one side of $S$, it does so on the other side, and the limiting values of the normal derivatives are equal. Gunther [6, p. 70] has quoted an example of a surface $S$ and continuous function $\nu$ such that $W[\nu]$ does not admit a regular normal derivative. Analytic conditions sufficient for the existence of a regular normal derivative $D\nu$ of $W[\nu]$ have been formulated by C. Neumann [12, p. 413], [13, p. 436], Liapounoff [8, p. 132], [9, §20, p. 293 et seq.], and Kellogg [7, p. 42 et seq.] while complicated necessary and sufficient conditions have been established by Petrini [15, p. 320], [16, p. 212]. Liapounoff [9, §19, p. 293] has characterized the domain of $D$ by showing that, for continuous $\nu$, $W[\nu]$ admits a regular normal derivative on $S$ when, and only when, the solution of the Dirichlet problem with respect to $R$, determined by the boundary values $\nu$, also admits a regular normal derivative on $S$.

Liapounoff [9, §§15, 16] (see also Plemelj [20, §4, p. 9]) has established the following extension of Green's third identity:

**Lemma.** Suppose that $V(P)$ is harmonic in $R$ and admits a regular normal derivative on $S$. Then, when $P \in R$,

\[(5) \quad \chi(P) = \frac{1}{2} W[V] - \frac{1}{2} V[\partial V/\partial n] = V(P)\]

and, when $P \in R'$, $\chi(P) = 0$.

It follows at once from (5), together with the formulae

\[(6) \quad W|_+ = \nu + KT\nu, \quad W|_- = -\nu + KT\nu,\]

describing the discontinuity in $W[\nu]$ across $S$, that, on $S$,

\[(7) \quad V = KTV - G(\partial V/\partial n).\]

Moreover, if it is assumed that $W[V]$ admits a regular normal derivative on $S$, it follows from (5) and the formulae

\[(8) \quad (\partial V/\partial n)|_+ = -\mu + K\mu, \quad (\partial V/\partial n)|_- = \mu + K\mu\]

describing the discontinuity in the normal derivative of $V[\mu]$ across $S$ that, on $S$,

\[(9) \quad \partial V/\partial n = DV - K(\partial V/\partial n).\]
The formulae (7) and (9) may be interpreted as integral equations connecting the limiting values \( V \) and \( \partial V/\partial n \) on \( S \). It is important to observe that (7) is the Neumann-Poincaré integral equation in the case \( \lambda = +1 \), corresponding to the exterior Dirichlet problem, while (9) is the Robin-Poincaré integral equation in the case \( \lambda = -1 \), corresponding to the exterior Neumann problem. The questions of existence and uniqueness of solutions of these equations have been discussed in detail by Plemelj [19, §16, p. 383 et seq.]. In particular, it is known that \( \lambda = -1 \) is not an eigenvalue of \( K \) and that (9) possesses an unique, continuous solution for any nonhomogeneous term \( D V \).

3. The characteristic properties of \( G \) and \( D \) now follow at once. For, whenever \( \mu \) is continuous on \( S \), substitution from the first of equations (8) into (7) is permissible and leads directly to the formula (see also Plemelj, loc. cit.)

\[
(10) \quad GK\mu = K^T G\mu. 
\]

Similarly, since \( V[\mu] \) admits a regular normal derivative on \( S \), so does \( W[V] \), and substitution from the first of equations (8) into (9) is also permissible, to obtain

\[
(11) \quad DG\mu = K^2 \mu - \mu. 
\]

When \( \nu \) is continuous on \( S \), \( W[\nu] \) may be represented in \( R \) as the sum \( W[\nu] = V_1(P) + V_2(P) \) of two harmonic functions characterized by the boundary values \( \nu \) and \( K^T \nu \) on \( S \), respectively. When \( W[\nu] \) admits a regular normal derivative on \( S \), so does \( V_1(P) \), whence \( V_2(P) \) does also, and so \( W[K^T \nu] \) does also. Thus \( D \) and \( DK^T \nu \) both exist, and \( D(\nu + K^T \nu) = D\nu + DK^T \nu \). Substituting, then, from the first of equations (6) into (9), and applying this relation, the formula

\[
(12) \quad DK^T \nu = K D\nu 
\]

is obtained. Similar substitution into (7) leads to

\[
(13) \quad GD\nu = (K^T)^2 \nu - \nu. 
\]

4. It is well known (Plemelj, loc. cit. §2) that \( \lambda = +1 \) is an eigenvalue of the kernel \( K \) of the Fredholm-Poincaré integral equations, and that, correspondingly, the homogeneous equations \( K^T \nu_1 - \nu_1 = 0 \), \( K\mu_1 - \mu_1 = 0 \) each admit a single eigenfunction. The eigenfunction \( \nu_1 \) is constant, while \( \mu_1 \) represents the equilibrium distribution of charge on \( S \). It follows from (10) that, with appropriate normalization, \( \nu_1 = G\mu_1 \). On the other hand, since \( \mu_1 \) is continuous, \( V[\mu_1] \) has a regular normal derivative on \( S \) whence, \( W[V] = W[\nu_1] \) has also. However, \( D\nu_1 = 0 \neq \mu_1 \).
The identities $K^2\mu - \mu = K\mu - \mu + K(K\mu - \mu)$ and $(K^T)^2\nu - \nu = K^T\nu - \nu + K^T(K^T\nu - \nu)$, together with the fact that $\lambda = -1$ is not an eigenvalue of $K$ or $K^T$, show that $\mu_1$ and $\nu_1$ are also the only eigenfunctions of $K^2$ and $(K^T)^2$ respectively. Thus, it follows from (13) that $D\nu = 0$ implies $\nu = \text{constant}$.

These remarks, together with Fredholm's third theorem, show that the integral equations (3) and (4) admit solutions when, and only when, $\int Df dS = 0$ and $\int gGdS = 0$, respectively; and that these solutions are not unique but contain an added arbitrary multiple of the corresponding eigenfunction.

5. Theorem 1. A necessary and sufficient condition that the integral equation (1) shall admit an unique continuous solution $\mu$ for any given continuous function $f$ is that $f$ lie in the domain of $D$. When this is the case, $\mu$ satisfies the equation (3).

Proof. 1. When $\mu$ is a continuous solution of (1), $V[\mu]$ admits a regular normal derivative on $S$ and, thus, so does $W[f]$. From (8) and (9) it follows that $f$ satisfies (3).

2. When $Df$ exists, equation (3) may be formulated and, since $\int S Df dS = 0$ this equation admits a continuous solution $\mu = \mu_0 + C\mu_1$. For each such solution it follows from (1) that $DG\mu = Df$, whence $G\mu - f = G\mu_0 + CG\mu_1 - f$ is constant on $S$. But $G\mu_1 = \nu_1$ is itself constant, thus $C$ may be uniquely chosen such that (1) is satisfied. Q.E.D.

Theorem 2. A necessary and sufficient condition that the integral equation (2) shall admit a continuous solution $\nu$, unique to within an additive constant, for any given continuous function $g$, is

\begin{equation}
\int_S g dS = 0.
\end{equation}

When this is the case, $\nu$ satisfies the equation (4) or, alternatively, $\nu = G\mu$ where $\mu$ satisfies the equation (3) with $Df$ replaced with $g$.

Proof. 1. When $\nu_0$ satisfies (2), so that $D\nu_0 = g = \partial/\partial n W[\nu_0]$ then $g$ satisfies (14). Moreover, from (13), $G D\nu_0 = Gg = (K^T)^2\nu_0 - \nu_0$, whence $\nu_0$ satisfies (4). Since $\nu_0$ lies in the domain of $D$, $\nu_0 = G\mu$ for some continuous $\mu$, and $\mu$ satisfies (3) with $Df$ replaced with $g$, by Theorem 1. These same conclusions are clearly valid for $\nu = \nu_0 + C\nu_1$, which also satisfies (2).

2. Given a continuous function $g$ satisfying (14), it follows that $\int_{S^1} Gg dS = \int_{S^1} GdS = \nu_1 \int g dS = 0$, whence (4) possesses a continuous solution $\nu = \nu_0 + C\nu_1$. Similarly (3) with $Df$ replaced with $g$ possesses a continuous solution $\mu = \mu_0 + C\mu_1$, whence $G\mu = G\mu_0 + CG\mu_1 = G\mu_0 + C\nu_1$.
and it follows from (10) that $\nu = G\mu$ and that every solution of (4) has this form. Thus, $W[\nu]$ has a regular normal derivative on $S$, and from (13) it follows that $GD\nu = Gf$, whence $D\nu = f$. Q.E.D.

Since $W[\nu]$ is constant in $R$ and zero in $R'$ this theorem is in accordance with the known fact that the Neumann problem possesses an unique, regular solution in $R'$, but that the solution is only determined to within an additive constant in $R$.

A second integral equation may, in certain circumstances, be formulated for the Dirichlet problem as follows:

**Corollary.** Suppose that $\int f dS = 0$. Then, the solution $\mu$ of (1) and (3) may be written $\mu = D\nu$ where $\nu$ is a continuous solution of (4) with $Gg$ replaced with $f$.

**Proof.** 1. When $\mu$ satisfies (1), $0 = \int f dS = \int \mu G dS = \int \nu dS = \nu \int f dS$ so that, by Theorem 2, $D\nu = \mu$ possesses a continuous solution $\nu$. This function satisfies (4) with $Gg$ replaced with $f$.

2. Following the arguments of Theorem 2, it is seen that (4), with $Gg$ replaced with $f$, possesses continuous solutions $\nu$ for which $W[\nu]$ admits an unique normal derivative on $S$. From (13), $GD\nu = f$ whence $\mu = D\nu$ is the solution of (1). Q.E.D.

**References**


University of Ottawa