SHORTER NOTES

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A UNIQUENESS THEOREM FOR CERTAIN TWO-POINT BOUNDARY VALUE PROBLEMS

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We are here concerned with establishing uniqueness of solutions for the following two-point boundary value problem:

(1) \[ x'' = f(t, x, x'), \quad x(a) = A, \quad x(b) = B, \]

with

(2) \[ f(t, x, u) - f(t, y, v) > g(t, x - y, u - v) \quad \text{if} \quad x > y, \]

where \( g(t, z, p) \) satisfies the following: (a) the initial value problem:

(3) \[ z'' = g(t, z, z'), \quad z(c) = 0, \quad z'(c) = C, \]

where \( c \geq a \) and \( C \) arbitrary, has a solution defined for all \( t \geq c \), (b) there exists a number \( h > 0 \) such that no nontrivial solution \( z(t) \) of (3) may satisfy \( z(c) = z(d) = 0 \), with \( d - c < h \), and (c) for any \( p \),

(4) \[ g(t, z_1, p) \geq g(t, z_2, p) \quad \text{if} \quad z_1 \geq z_2. \]

Our main result is the following:

Theorem. Under the above assumptions, if \( b - a \leq h \), then (1) has at most one solution.

Proof. Suppose that \( x_1(t) \) and \( x_2(t) \) are two distinct solutions of (1), and write \( \psi(t) = x_1(t) - x_2(t) \). Without loss of generality, we may assume that there exist numbers \( c, d \) such that \( a \leq c < d \leq b \), \( \psi(c) = \psi(d) = 0 \) and \( \psi(t) > 0 \) for \( t \in (c, d) \). Consider the solution of (3) with initial conditions \( z(c) = 0, z'(c) = \phi'(c) \). Let \( \psi(t) = \phi(t) - z(t) \). Clearly, \( \psi(c) = \psi'(c) = 0 \). From (1), (2), and (3), observe that

\[
\psi''(c) = \phi''(c) - z''(c) \\
= x_1''(c) - x_2''(c) - z''(c) \\
= f(c, x_1(c), x_1'(c)) - f(c, x_2(c), x_2'(c)) - z''(c) \\
> g(c, 0, x_1'(c) - x_2'(c)) - g(x, 0, z'(c)) \\
= g(c, 0, \phi'(c)) - g(c, 0, z'(c)) = 0,
\]

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hence one concludes that there exists $t_0 \in (c, d)$ such that $\psi(c) = \psi(t_0) = 0$ and $\psi(t) > 0$ for $t \in (c, t_0)$. Hence there must exist a number $t_1 \in (c, t_0)$ such that $\psi'(t_1) = 0$, and $\psi''(t_1) \leq 0$. On the other hand, we note

$$\phi''(t_1) - z''(t_1) > g(t_1, \phi(t_1), \phi'(t_1)) - g(t_1, z(t_1), z'(t_1)) \geq 0,$$

which is the desired contradiction.

In the special case when $f(t, x, x') = h(t) - k(x)$ and $g(t, y, y') = -y$ for some continuous functions $h(t)$ and $k(x)$, the above theorem simplifies the hypothesis and the proof of a recent result of Marie and Tomic [1] on the Duffing problem. Our result is somewhat related to the classical problem of estimating the interval of uniqueness of (1). In contrast to the traditional approach, we impose a strict inequality (namely (2)) and obtain a larger interval of uniqueness (i.e., $b - a < h$). On the other hand, the traditional approach when $g(t, y, p)$ is linear in $y$ and $p$, yields $b - a < h$ which does not cover the result of [1]. The fact that the one-sided inequality (2) plays an important role in the uniqueness theorem for (1) has also been explored by Bailey and Waltman [2], where other references may be found.

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ADDED IN PROOF. Cf. also, D. Willett, SIAM Review 9 (1967), 726–728.

REFERENCES


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