GENERALIZED EIGENFUNCTIONS OF THE LAPLACE OPERATOR AND WEIGHTED AVERAGE PROPERTY

ANIL KUMAR BOSE

1. Introduction. In the previous two papers, [1] and [2], we were interested in characterizing the class $S(w, R)$ of real-valued functions $u$, defined in a given region (open, connected set) $R$ of the $n$-dimensional Euclidean space $E_n$ which satisfy the Weighted Average Property (W.A.P.):

$$u(P) = \frac{\int_{B(P, r)} u \cdot w \, dp}{\int_{B(P, r)} w \, dp}, \quad P \in R,$$

where $B(P, r)$ denotes any ball with the point $P = (x_1, x_2, \cdots, x_n)$ for its center and radius $r$ whose closure lies in $R$; $dp$ stands for the usual Lebesgue measure of $B$ and $w$ is a weight function (W.F.) defined in $R$ (i.e., $w$ is nonnegative and locally summable in $R$).

It was proved in paper [1] that “$S(w, R)$ is always a subspace of the solution space of the second-order linear elliptic homogeneous differential equation:

$$w \Delta u + 2 \sum_{i=1}^{n} u_{x_i} w_{x_i} = 0$$

in $R$, where $u$ is the Laplacian of $u$, provided $w \in C^1(R)$.” Furthermore, “$S(w, R)$ is the solution space of the equation (2) if the W.F. $w$ is an eigenfunction of the Laplace operator. That is, if $w$ is a solution of an equation of the form:

$$\Delta w + \lambda w = 0$$

in $R$, where $\lambda$ is some real constant.” In the latter case “$S(w, R)$ is infinite dimensional.”

The present paper is an extension of these results. In this paper we want to prove that “$S(w, R)$ is always a subspace of the (common) solution space of a system of equations of the form:

$$\Delta^k w \cdot \Delta u + 2 \sum_{i=1}^{n} u_{x_i} (\Delta^k w)_{x_i} = 0, \quad k = 0, 1, 2, 3, \cdots,$$

where $\Delta^k w$ is the $k$th iteration of the Laplacian of $w$ ($\Delta^0 w$ is interpreted as $w$), provided the weight function $w$ is sufficiently differentiable.”

Presented to Society, January 27, 1967; received by the editors December 12, 1966.
Furthermore, "S(w, R) is precisely the solution space of the system of equations (4), provided the weight function w is a generalized eigenfunction of the Laplace operator or more generally a weight function satisfying an equation of the form:

$$\Delta^m w = a_0 w + a_1 \Delta w + \cdots + a_{m-1} \Delta^{m-1} w,$$

in the region R, where m is a positive integer and the a_i's are real constants."

In paper [2] it was proved that "in $E_2$, S(w, R) is infinite dimensional if and only if w is an eigenfunction of the Laplace operator; otherwise, S(w, R) is finite dimensional and $1 \leq \dim S(w, R) \leq 2$." It was also claimed there that in $E_n, n > 2$, "S(w, R) is infinite dimensional if and only if w is an eigenfunction of the Laplace operator; otherwise, S(w, R) is finite dimensional and $1 \leq \dim S(w, R) \leq 2n - 1." In a footnote of [2] it was mentioned that the above statement is, possibly, not true in $E_n, n > 2$. The main result of this paper clarifies this point completely. Indeed, it will be shown in §III of this paper that in $E_n, n > 2$, S(w, R) could be infinite dimensional even if the weight function w is not an eigenfunction of the Laplace operator.

II. Theorem 1. Let R be a region in $E_n$ and w be a W. F. defined in R.

(i) If $w \in C^{2m+1}(R)$, m being a nonnegative integer, then S(w, R) is a subspace of the solution space of the system of equations:

$$\Delta^k w \cdot \Delta u + 2 \sum_{i=1}^{n} u_{x_i}(\Delta^k w)_{x_i} = 0, \text{ in } R, k = 0, 1, 2, \cdots, m.$$

(ii) If $\Delta^k w$ is defined and is in $C^1(R)$ for each positive integer k, then S(w, R) is a subspace of the solution space of the system of equations:

$$\Delta^k w \cdot \Delta u + 2 \sum_{i=1}^{n} u_{x_i}(\Delta^k w)_{x_i} = 0, \text{ in } R, k = 0, 1, 2, 3, \cdots, \infty$$

Proof. Part (i). Let $u \in S(w, R)$. Then by Theorem 2 and Theorem 4 of [1], $u \in C^{2m+2}(R)$ and

$$w \Delta w + 2 \sum_{i=1}^{n} u_{x_i} w_{x_i} = 0 \text{ in } R.$$

Also by Theorem 3 of [1], we get the circumferential mean-value property

$$\int_{S(P,r)} uw \, d\sigma = u(P) \int_{S(P,r)} w \, d\sigma,$$
for each \( S(P, r) \)—the boundary of \( B(P, r) \)—which, together with its interior \( B(P, r) \), lies in \( R \). As proved in Theorem 4 of [1], we get immediately from (8), by differentiating with respect to \( r \), the mean-
value relation

\[
\frac{\partial w}{\partial n} = \int_{\partial S(P, r)} \frac{\partial u}{\partial n} \, d\sigma
\]

for each \( S(P, r) \) which, together with its interior \( B(P, r) \), lies in \( R \), where \( \partial / \partial n \) refers to the derivative in the direction of the outward drawn normal to the surface \( S(P, r) \). Now, using Green's formula and relations (7) and (9), we get

\[
\int_{B(P, r)} \left( w \Delta u + 2 \sum_{i=1}^{n} u_{x_i} w_{x_i} + u \Delta w \right) \, d\rho = \int_{B(P, r)} \Delta (uw) \, d\rho
\]

\[
= \int_{S(P, r)} \frac{\partial uw}{\partial n} \, d\sigma = u(P) \int_{S(P, r)} \frac{\partial w}{\partial n} \, d\sigma
\]

or

\[
\int_{B(P, r)} u \Delta wd\rho = u(P) \int_{B(P, r)} \Delta wd\rho,
\]

for each \( B(P, r) \) whose closure lies in \( R \). It is now clear that we can apply the entire reasoning of Theorem 4 of [1] over again to the averaging property (10) and get

\[
\Delta w \cdot \Delta u + 2 \sum_{i=1}^{n} u_{x_i} (\Delta w)_{x_i} = 0 \quad \text{in } R.
\]

Since \( \Delta^k w \) is defined and is in class \( C^1(R) \) for \( k = 0, 1, 2, \ldots, m \), repeating the argument a finite number of times, we see that \( u \) must satisfy the system of equations:

\[
\Delta^k w \cdot \Delta u + 2 \sum_{i=1}^{n} u_{x_i} (\Delta^k w)_{x_i} = 0, \quad k = 0, 1, 2, \ldots, m \quad \text{in } R.
\]

This proves Part (i).

Part (ii). Let \( u \in S(w, R) \). Since \( \Delta^k w \) is defined and is in class \( C^1(R) \) for all \( k = 0, 1, 2, 3, \ldots \), the above reasoning together with mathematical induction implies that \( u \) must be a common solution of the system of equations:

\[
\Delta^k w \cdot \Delta u + 2 \sum_{i=1}^{n} u_{x_i} (\Delta^k w)_{x_i} = 0
\]
in $R$, $k = 0, 1, 2, 3, \ldots, \infty$. This proves Part (ii).

**Theorem 2.** Defined in a region $R$ of $E_n$, let $w$ be a W. F. belonging to class $C^{2m}(R)$ and be a solution of the differential equation

$$
\Delta^m w = a_0 w + a_1 \Delta w + \cdots + a_{m-1} \Delta^{m-1} w
$$

in $R$, where $m$ is a positive integer and the $a_i$'s are real constants. A real-valued function $u$ is in $S(w, R)$ if and only if $u$ is in $C^2(R)$ and is a common solution of the system of equations:

$$
\Delta^k u + 2 \sum_{i=1}^{n} u_{x_i} (\Delta^k w)_{x_i} = 0, \quad k = 0, 1, 2, \ldots, m - 1, \quad in \ R.
$$

**Proof.** Let $u \in S(w, R)$. By Theorem 2 of [1], $u \in C^{2m+1}(R)$. Since $w \in C^{2m}(R)$, the right-hand side of (12) is in $C^2(R)$. Hence, $\Delta^{m+1} w$ is defined in $R$ and is a linear combination of $w, \Delta w, \Delta^2 w, \cdots, \Delta^{m-1} w$ in $R$. Applying mathematical induction, it is easy to see that $\Delta^{m+p} w$ is defined in $R$ and $\Delta^{m+p} w$ is a linear combination of $w, \Delta w, \Delta^2 w, \cdots, \Delta^{m-1} w$ in $R$ for each positive integer $p$. This means that $\Delta^k w$ is defined in $R$ and is a linear combination of $w, \Delta w, \Delta^2 w, \cdots, \Delta^{m-1} w$ in $R$ for each positive integer $k$. Hence, by Part (ii) of Theorem 1, it follows that $u$ is a common solution of the system of equations

$$
\Delta^k w \cdot \Delta u + 2 \sum_{i=1}^{n} u_{x_i} (\Delta^k w)_{x_i} = 0
$$

in $R$, $k = 0, 1, 2, 3, \ldots, \infty$. Hence, $u$ is a common solution of system (13) of equations.

Conversely, suppose that $u \in C^2(R)$ and is a common solution of the system of equations (13) in $R$. Since $\Delta^k w$ is defined in $R$ and is a linear combination of $w, \Delta w, \Delta^2 w, \cdots, \Delta^{m-1} w$ for each positive integer $k$, it follows that every member of the system (14) of equations can be expressed as a linear combination of the $m$ equations of the system (13). That is, $u$ is also a common solution of the system (14) of equations in $R$. Since $\Delta(uw)$ is defined in $R$ and $u$ is a solution of

$$
w \Delta u + 2 \sum_{i=1}^{n} u_{x_i} w_{x_i} = 0 \quad in \ R,
$$

we have

$$
\Delta(uw) = w \Delta u + 2 \sum_{i=1}^{n} u_{x_i} w_{x_i} + u \Delta w = u \Delta w \quad in \ R.
$$
Now, assuming that $\Delta^k(uw)$ is defined and $\Delta^k(uw) = u\Delta^k w$ in $R$ for some positive integer $k$ and using the fact that $u$ is a solution of the system (14) of equations in $R$, we see that $\Delta^{k+1}(uw)$ is defined in $R$ and

$$\Delta^{k+1}(uw) = \Delta^k w \cdot \Delta u + 2 \sum_{i=1}^{n} u_{zi}(\Delta^k w)_{x_i} + u \Delta^{k+1} w = u \Delta^{k+1} w \text{ in } R.$$ 

Hence, by mathematical induction $\Delta^k(uw)$ is defined in $R$ and

$$\Delta^k(uw) = u \cdot \Delta^k w$$

in $R$ for each positive integer $k$. Putting $k = m$ and using (12), we see that

$$\Delta^m(uw) = u \Delta^m w = a_0(uw) + a_1 \Delta(uw) + \cdots + a_{m-1} \Delta^{m-1}(uw).$$

Therefore, $uw$ is also a solution of equation (12) in $R$. Let $B(P, r)$ be any ball which, together with its boundary $S(P, r)$, lies in $R$. As proved in [3, pp. 286–289],

$$\frac{1}{\Omega_r} \int_{S(P, r)} \omega \, d\sigma = \Gamma(n/2) \sum_{r=0}^{m+k-1} \frac{(r/2)^{2r}}{\nu! \Gamma(\nu + n/2)} \frac{\Delta^r w(P)}{\nu! \Gamma(\nu + n/2)}$$

$$+ \int_{B(P, r)} v_{m+k}(\beta) \Delta^{m+k} w \, d\rho$$

for all positive integers $k$, where $\Omega_r$ is the surface area of $S(P, r)$ and the sequence of functions $v_m(\beta)$ are given by the recursion system

$$v_{r+1}(\beta) = \frac{1}{(n-2)\beta^{n-2}} \int_{\beta}^{r} \alpha v_r(\alpha) (\alpha^{n-2} - \beta^{n-2}) \, d\alpha,$$

$$v_0(\beta) = (1/(n-2)\Omega_1)(1/\beta^{n-2} - 1/r^{n-2}),$$

$\Omega_1$ being the surface area of the unit sphere in $n$-dimensional space and $\beta$ being the distance of a point in $B(P, r)$ from the center $P$. [In two dimensions the recursion formula (17) is given by

$$v_{r+1}(\beta) = \int_\beta^r \alpha v_r(\alpha) \log \alpha/\beta \, d\alpha,$$

$$v_0(\beta) = (1/2\pi) \log r/\beta.]$$

Since $\Delta^m w = a_0 w + a_1 \Delta w + \cdots + a_{m-1} \Delta^{m-1} w$, we have

$$\Delta^{m+k} w = a_0^{(k)} w + a_1^{(k)} \Delta w + \cdots + a_{m-1}^{(k)} \Delta^{m-1} w, \quad k = 1, 2, 3, \ldots, \infty,$$

where the sequences of constants $\{a_i^{(k)}\}$, $i = 0, 1, 2, \ldots, m-1$, are defined recursively by
\[ a_0^{(k)} = a_{m-1} a_0, \]
\[ a_1^{(k)} = a_0^{(k-1)} a_1 + a_{m-1} a_0^{(k-1)} a_1 + \cdots + a_{m-2} a_1 + a_{m-1} a_0^{(k-1)} a_1, \]
a\[ a_i^{(0)} \] being interpreted as \( a_i, i = 0, 1, \ldots, m-1. \]

Let \( c/2 \) be a positive number greater than each of the numbers 1, \(|a_0|, |a_1|, \ldots, |a_{n-1}|. \) Then it is easy to see that

\[ |a_i^{(k)}| < c \]

for all \( i = 0, 1, 2, \ldots, m-1, \) and for all nonnegative integers \( k. \) Also, the \( m \) functions \( w, \Delta w, \Delta^2 w, \ldots, \Delta^{m-1} w \) are each bounded by a positive constant \( \lambda \) on the closure of \( B(P, r). \) Now, it has already been shown in \([3, \text{ p. 289}]\) that the remainder term \( \int_{B(P, r)} v_{m+k} \Delta^{m+k} w dp \) in (16) tends to zero as \( k \) tends to \( \infty, \) provided \( w \in C^2(R) \) and is a solution of \( \Delta w - cw = 0 \) in \( R, \) where \( c \) is a positive constant. Since \( g = \exp(c/n)^{1/2} \sum_{i=1}^n x_i \) is in \( C^2(R) \) and satisfies \( \Delta w - cw = 0 \) in \( R, \) where \( c \) is given by (19), we have

\[ \lim_{k \to \infty} \int_{B(P, r)} v_{m+k} \Delta^{m+k} g dp = 0. \]

Let \( g_0 \) be the minimum of \( g \) on the closure of \( B(P, r). \) Then \( g_0 > 0 \) and

\[ 0 \leq c^{m+k} g_0 \int_{B(P, r)} v_{m+k} dp \leq \int_{B(P, r)} v_{m+k} c^{m+k} g dp \]
\[ = \int_{B(P, r)} v_{m+k} \Delta^{m+k} g dp. \]

From (20) and (21) we see that

\[ \lim_{k \to \infty} c^{m+k} \int_{B(P, r)} v_{m+k} dp = 0. \]

Now, returning to the relation (16), where \( w \) satisfies the equation (12), we see that

\[ 0 \leq \left| \int_{B(P, r)} v_{m+k} \Delta^{m+k} w dp \right| \leq \int_{B(P, r)} v_{m+k} \Delta^{m+k} w dp \]
\[ \leq (m\lambda/c^{m-1}) c^{m+k} \int_{B(P, r)} v_{m+k} dp. \]

From (22) and (23) it now follows that the remainder term \( \int_{B(P, r)} v_{m+k} \Delta^{m+k} w dp \) in (16) tends to zero as \( k \) tends to \( \infty. \) Hence, we have
\begin{align*}
(24) \quad \frac{1}{\Omega_r} \int_{S(P,r)} w \, d\sigma &= \Gamma(n/2) \sum_{r=0}^{\infty} \left( \frac{r/2}{v} \right)^{2v} \frac{\Delta^r w(P)}{v! \Gamma(v + n/2)}.

\text{Similarly,}

(25) \quad \frac{1}{\Omega_r} \int_{S(P,r)} u \, d\omega &= \Gamma(n/2) \sum_{r=0}^{\infty} \left( \frac{r/2}{v} \right)^{2v} \frac{w(P) \Delta^r w(P)}{v! \Gamma(v + n/2)}.

\text{From (24) and (25) we get}

(26) \quad \int_{S(P,r)} u \, d\omega &= u(P) \int_{S(P,r)} w \, d\sigma,

\text{for each } S(P, r) \text{ which together with its interior } B(P, r) \text{ lies in } R. \text{ This means by Theorem 3 of [1] that } u \in S(w, R). \text{ This completes the proof.}

\text{III. As mentioned in the Introduction, we will now give an example to show that in } E_n, n > 2, S(w, R) \text{ can be infinite dimensional even if the W. F. } w \text{ is not an eigenfunction of the Laplace operator.}

\text{In } E_3 \text{ let us consider the following W. F. :}

w(x_1, x_2, x_3) = e^{x_1} + e^{x_2}, \text{ for all } (x_1, x_2, x_3) \in E_3.

\text{Clearly, } w \text{ is not an eigenfunction of the Laplace operator. We have}

(27) \quad \Delta^2 w = 5\Delta w - 4w, \quad w_{x_1} = w_{x_2} = (\Delta w)_{x_1} = (\Delta w)_{x_2} = 0

\text{in } E_3. \text{ Hence, by Theorem 2, } S(w, E_3) \text{ is the solution space of the system of elliptic equations:}

\begin{align*}
(28) \quad w\Delta u &+ 2 \sum_{i=1}^{3} u_{x_i} w_{x_i} = 0, \\
\Delta w \cdot \Delta u &+ 2 \sum_{i=1}^{3} u_{x_i} (\Delta w)_{x_i} = 0
\end{align*}

\text{in } E_3.

\text{Now, for each positive integer } k, \text{ let } P_k(x_1, x_2) \text{ be a harmonic polynomial of degree } k \text{ in } E_2. \text{ Then each of the functions}

f_k(x_1, x_2, x_3) = P_k(x_1, x_2), \quad k = 1, 2, 3, \cdots, \infty,

\text{is defined in } E_3 \text{ and is harmonic in } E_3. \text{ Hence,}

\Delta f_k = 0 = (f_k)_{x_3}

\text{in } E_3, \text{ for all positive integers } k. \text{ It is clear that each of the functions } f_k \text{ is a solution of system (24) of equations. Therefore, } f_k \in S(w, E_3)
for each positive integer \( k \). Since the infinite set of functions \( \{f_1, f_2, \ldots, f_k, \ldots \} \) is linearly independent over \( E_3 \), it follows that \( S(w, E_3) \) is infinite dimensional.

It is also clear that similar types of examples can be constructed in four- and higher-dimensional Euclidean spaces.

**Bibliography**


**University of Alabama**