GENERALIZED EIGENFUNCTIONS OF THE LAPLACE OPERATOR AND WEIGHTED AVERAGE PROPERTY

ANIL KUMAR BOSE

I. **Introduction.** In the previous two papers, [1] and [2], we were interested in characterizing the class $S(w, R)$ of real-valued functions u, defined in a given region (open, connected set) R of the n-dimensional Euclidean space E_n which satisfy the Weighted Average Property (W.A.P.):

$$(1) \qquad u(P) = \int_{B(P,r)} u \cdot w d\rho \Big/ \int_{B(P,r)} w d\rho, \qquad P \in R,$$

where $B(P, r)$ denotes any ball with the point $P=P(x_1, x_2, \cdots, x_n)$ for its center and radius r whose closure lies in R; $d\rho$ stands for the usual Lebesgue measure of B and w is a weight function (W.F.) defined in R (i.e., w is nonnegative and locally summable in R).

It was proved in paper [1] that "$S(w, R)$ is always a subspace of the solution space of the second-order linear elliptic homogeneous differential equation:

$$(2) \qquad w\Delta u + 2 \sum_{i=1}^{n} u_{x_i} w_{x_i} = 0$$

in R, where u is the Laplacian of u, provided $w \in C^1(R)$." Furthermore, "$S(w, R)$ is the solution space of the equation (2) if the W.F. w is an eigenfunction of the Laplace operator. That is, if w is a solution of an equation of the form:

$$(3) \qquad \Delta w + \lambda w = 0$$

in R, where λ is some real constant." In the latter case "$S(w, R)$ is infinite dimensional."

The present paper is an extension of these results. In this paper we want to prove that "$S(w, R)$ is always a subspace of the (common) solution space of a system of equations of the form:

$$(4) \qquad \Delta^k w \cdot \Delta u + 2 \sum_{i=1}^{n} u_{x_i} (\Delta^k w)_{x_i} = 0, \qquad k = 0, 1, 2, 3, \cdots,$$

where $\Delta^k w$ is the kth iteration of the Laplacian of w ($\Delta^0 w$ is interpreted as w), provided the weight function w is sufficiently differentiable."

Presented to Society, January 27, 1967; received by the editors December 12, 1966.

Furthermore, "$S(w, R)$ is precisely the solution space of the system of equations (4), provided the weight function w is a generalized eigenfunction of the Laplace operator or more generally a weight function satisfying an equation of the form:

$$\Delta^m w = a_0 w + a_1 \Delta w + \cdots + a_{m-1}\Delta^{m-1} w, \tag{5}$$

in the region R, where m is a positive integer and the a_i's are real constants."

In paper [2] it was proved that "in E_2, $S(w, R)$ is infinite dimensional if and only if w is an eigenfunction of the Laplace operator; otherwise, $S(w, R)$ is finite dimensional and $1 \leqq \dim S(w, R) \leqq 2$." It was also claimed there that in E_n, $n>2$, "$S(w, R)$ is infinite dimensional if and only if w is an eigenfunction of the Laplace operator; otherwise, $S(w, R)$ is finite dimensional and $1 \leqq \dim S(w, R) \leqq 2n-1$." In a footnote of [2] it was mentioned that the above statement is, possibly, not true in E_n, $n>2$. The main result of this paper clarifies this point completely. Indeed, it will be shown in §III of this paper that in E_n, $n>2$, $S(w, R)$ could be infinite dimensional even if the weight function w is not an eigenfunction of the Laplace operator.

II. THEOREM 1. *Let R be a region in E_n and w be a W. F. defined in R.*

(i) *If $w \in C^{2m+1}(R)$, m being a nonnegative integer, then $S(w, R)$ is a subspace of the solution space of the system of equations*:

$$\Delta^k w \cdot \Delta u + 2 \sum_{i=1}^{n} u_{x_i}(\Delta^k w)_{x_i} = 0, \quad \text{in } R,\ k = 0, 1, 2, \cdots, m. \tag{6}$$

(ii) *If $\Delta^k w$ is defined and is in $C^1(R)$ for each positive integer k, then $S(w, R)$ is a subspace of the solution space of the system of equations*:

$$\Delta^k w \cdot \Delta u + 2 \sum_{i=1}^{n} u_{x_i}(\Delta^k w)_{x_i} = 0 \quad \text{in } R,\ k = 0, 1, 2, 3, \cdots, \infty$$

PROOF. Part (i). Let $u \in S(w, R)$. Then by Theorem 2 and Theorem 4 of [1], $u \in C^{2m+2}(R)$ and

$$w \Delta w + 2 \sum_{i=1}^{n} u_{x_i} w_{x_i} = 0 \quad \text{in } R. \tag{7}$$

Also by Theorem 3 of [1], we get the circumferential mean-value property

$$\int_{S(P,r)} uw\,d\sigma = u(P) \int_{S(P,r)} w\,d\sigma, \tag{8}$$

for each $S(P, r)$—the boundary of $B(P, r)$—which, together with its interior $B(P, r)$, lies in R. As proved in Theorem 4 of [1], we get immediately from (8), by differentiating with respect to r, the mean-value relation

$$u(P)\int_{S(P,r)} \frac{\partial w}{\partial n}\,d\sigma = \int_{S(P,r)} \frac{\partial uw}{\partial n}\,d\sigma \tag{9}$$

for each $S(P, r)$ which, together with its interior $B(P, r)$, lies in R, where $\partial/\partial n$ refers to the derivative in the direction of the outward drawn normal to the surface $S(P, r)$. Now, using Green's formula and relations (7) and (9), we get

$$\int_{B(P,r)}\left(w\Delta u + 2\sum_{i=1}^{n} u_{x_i}w_{x_i} + u\Delta w\right)d\rho = \int_{B(P,r)} \Delta(uw)\,d\rho$$
$$= \int_{S(P,r)} \frac{\partial uw}{\partial n}\,d\sigma = u(P)\int_{S(P,r)} \frac{\partial w}{\partial n}\,d\sigma$$

or

$$\int_{B(P,r)} u\Delta w\,d\rho = u(P)\int_{B(P,r)} \Delta w\,d\rho, \tag{10}$$

for each $B(P, r)$ whose closure lies in R. It is now clear that we can apply the entire reasoning of Theorem 4 of [1] over again to the averaging property (10) and get

$$\Delta w\cdot\Delta u + 2\sum_{i=1}^{n} u_{x_i}(\Delta w)_{x_i} = 0 \quad \text{in } R. \tag{11}$$

Since $\Delta^k w$ is defined and is in class $C^1(R)$ for $k=0, 1, 2, \cdots, m$, repeating the argument a finite number of times, we see that u must satisfy the system of equations:

$$\Delta^k w\cdot\Delta u + 2\sum_{i=1}^{n} u_{x_i}(\Delta^k w)_{x_i} = 0, \qquad k = 0, 1, 2, \cdots, m \quad \text{in } R.$$

This proves Part (i).

Part (ii). Let $u\in S(w, R)$. Since $\Delta^k w$ is defined and is in class $C^1(R)$ for all $k=0, 1, 2, 3, \cdots$, the above reasoning together with mathematical induction implies that u must be a common solution of the system of equations:

$$\Delta^k w\cdot\Delta u + 2\sum_{i=1}^{n} u_{x_i}(\Delta^k w)_{x_i} = 0$$

in R, $k=0, 1, 2, 3, \cdots, \infty$. This proves Part (ii).

THEOREM 2. *Defined in a region R of E_n, let w be a W. F. belonging to class $C^{2m}(R)$ and be a solution of the differential equation*

$$\Delta^m w = a_0 w + a_1 \Delta w + \cdots + a_{m-1}\Delta^{m-1} w \tag{12}$$

in R, where m is a positive integer and the a_i's are real constants. A real-valued function u is in $S(w, R)$ if and only if u is in $C^2(R)$ and is a common solution of the system of equations:

$$\Delta^k w \cdot \Delta u + 2 \sum_{i=1}^{n} u_{x_i}(\Delta^k w)_{x_i} = 0, \quad k = 0, 1, 2, \cdots, m-1, \quad \textit{in } R. \tag{13}$$

PROOF. Let $u \in S(w, R)$. By Theorem 2 of [1], $u \in C^{2m+1}(R)$. Since $w \in C^{2m}(R)$, the right-hand side of (12) is in $C^2(R)$. Hence, $\Delta^{m+1}w$ is defined in R and is a linear combination of $w, \Delta w, \Delta^2 w, \cdots, \Delta^{m-1}w$ in R. Applying mathematical induction, it is easy to see that $\Delta^{m+p}w$ is defined in R and $\Delta^{m+p}w$ is a linear combination of $w, \Delta w, \Delta^2 w, \cdots, \Delta^{m-1}w$ in R for each positive integer p. This means that $\Delta^k w$ is defined in R and is a linear combination of $w, \Delta w, \Delta^2 w, \cdots, \Delta^{m-1}w$ in R for each positive integer k. Hence, by Part (ii) of Theorem 1, it follows that u is a common solution of the system of equations

$$\Delta^k w \cdot \Delta u + 2 \sum_{i=1}^{n} u_{x_i}(\Delta^k w)_{x_i} = 0 \tag{14}$$

in R, $k=0, 1, 2, 3, \cdots, \infty$. Hence, u is a common solution of system (13) of equations.

Conversely, suppose that $u \in C^2(R)$ and is a common solution of the system of equations (13) in R. Since $\Delta^k w$ is defined in R and is a linear combination of $w, \Delta w, \Delta^2 w, \cdots, \Delta^{m-1}w$ for each positive integer k, it follows that every member of the system (14) of equations can be expressed as a linear combination of the m equations of the system (13). That is, u is also a common solution of the system (14) of equations in R. Since $\Delta(uw)$ is defined in R and u is a solution of

$$w\Delta u + 2 \sum_{i=1}^{n} u_{x_i} w_{x_i} = 0 \quad \text{in } R,$$

we have

$$\Delta(uw) = w\Delta u + 2 \sum_{i=1}^{n} u_{x_i} w_{x_i} + u\Delta w = u\Delta w \quad \text{in } R.$$

Now, assuming that $\Delta^k(uw)$ is defined and $\Delta^k(uw)=u\Delta^k w$ in R for some positive integer k and using the fact that u is a solution of the system (14) of equations in R, we see that $\Delta^{k+1}(uw)$ is defined in R and

$$\Delta^{k+1}(uw) = \Delta^k w\cdot \Delta u + 2\sum_{i=1}^{n} u_{x_i}(\Delta^k w)_{x_i} + u\Delta^{k+1}w = u\Delta^{k+1}w \quad \text{in } R.$$

Hence, by mathematical induction $\Delta^k(uw)$ is defined in R and

$$\Delta^k(uw) = u\cdot\Delta^k w \tag{15}$$

in R for each positive integer k. Putting $k=m$ and using (12), we see that

$$\Delta^m(uw) = u\Delta^m w = a_0(uw) + a_1\Delta(uw) + \cdots + a_{m-1}\Delta^{m-1}(uw).$$

Therefore, uw is also a solution of equation (12) in R. Let $B(P, r)$ be any ball which, together with its boundary $S(P, r)$, lies in R. As proved in [**3**, pp. 286–289].

$$\begin{aligned}\frac{1}{\Omega_r}\int_{S(P,r)} w\,d\sigma &= \Gamma(n/2)\sum_{\nu=0}^{m+k-1}(r/2)^{2\nu}\frac{\Delta^\nu w(P)}{\nu!\Gamma(\nu+n/2)}\\ &\quad + \int_{B(P,r)} v_{m+k}(\beta)\,\Delta^{m+k}w\,d\rho\end{aligned} \tag{16}$$

for all positive integers k, where Ω_r is the surface area of $S(P, r)$ and the sequence of functions $v_m(\beta)$ are given by the recursion system

$$\begin{aligned} v_{\nu+1}(\beta) &= \frac{1}{(n-2)\beta^{n-2}}\int_\beta^r \alpha v_\nu(\alpha)(\alpha^{n-2}-\beta^{n-2})\,d\alpha,\\ v_0(\beta) &= (1/(n-2)\Omega_1)(1/\beta^{n-2} - 1/r^{n-2}),\end{aligned} \tag{17}$$

Ω_1 being the surface area of the unit sphere in n-dimensional space and β being the distance of a point in $B(P, r)$ from the center P. [In two dimensions the recursion formula (17) is given by

$$\begin{aligned} v_{\nu+1}(\beta) &= \int_\beta^r \alpha v_\nu(\alpha)\log\alpha/\beta\,d\alpha,\\ v_0(\beta) &= (1/2\pi)\log r/\beta.]\end{aligned} \tag{18}$$

Since $\Delta^m w = a_0 w + a_1\Delta w + \cdots + a_{m-1}\Delta^{m-1}w$, we have

$$\Delta^{m+k} w = a_0^{(k)} w + a_1^{(k)}\Delta w + \cdots + a_{m-1}^{(k)}\Delta^{m-1} w, \qquad k = 1, 2, 3, \cdots, \infty,$$

where the sequences of constants $\{a_i^{(k)}\}$, $i=0, 1, 2, \cdots, m-1$, are defined recursively by

$$a_0^{(k)} = a_{m-1}^{(k-1)} a_0,$$
$$a_1^{(k)} = a_0^{(k-1)} + a_{m-1}^{(k-1)} a_1, \cdots, a_{m-1}^{(k)} = a_{m-2}^{(k-1)} + a_{m-1}^{(k-1)} a_{m-1},$$

$a_i^{(0)}$ being interpreted as a_i, $i=0, 1, \cdots, m-1$.

Let $c/2$ be a positive number greater than each of the numbers 1, $|a_0|, |a_1|, \cdots, |a_{n-1}|$. Then it is easy to see that

$$|a_i^{(k)}| < c^{k+1} \tag{19}$$

for all $i=0, 1, 2, \cdots, m-1$, and for all nonnegative integers k. Also, the m functions $w, \Delta w, \Delta^2 w, \cdots, \Delta^{m-1}w$ are each bounded by a positive constant λ on the closure of $B(P, r)$. Now, it has already been shown in [3, p. 289] that the remainder term $\int_{B(P,r)} v_{m+k}\Delta^{m+k}w d\rho$ in (16) tends to zero as k tends to ∞, provided $w \in C^2(R)$ and is a solution of $\Delta w - cw = 0$ in R, where c is a positive constant. Since $g = \exp(c/n)^{1/2} \sum_{i=1}^n x_i)$ is in $C^2(R)$ and satisfies $\Delta w - cw = 0$ in R, where c is given by (19), we have

$$\lim_{k\to\infty} \int_{B(P,r)} v_{m+k}\Delta^{m+k} g d\rho = 0. \tag{20}$$

Let g_0 be the minimum of g on the closure of $B(P, r)$. Then $g_0 > 0$ and

$$\begin{aligned} 0 &\leqq c^{m+k} g_0 \int_{B(P,r)} v_{m+k} d\rho \leqq \int_{B(P,r)} v_{m+k} c^{m+k} g d\rho \\ &= \int_{B(P,r)} v_{m+k}\Delta^{m+k} g d\rho. \end{aligned} \tag{21}$$

From (20) and (21) we see that

$$\lim_{k\to\infty} c^{m+k} \int_{B(P,r)} v_{m+k} d\rho = 0. \tag{22}$$

Now, returning to the relation (16), where w satisfies the equation (12), we see that

$$\begin{aligned} 0 &\leqq \left| \int_{B(P,r)} v_{m+k}\Delta^{m+k} w d\rho \right| \leqq \int_{B(P,r)} v_{m+k} \left| \Delta^{m+k} w \right| d\rho \\ &\leqq (m\lambda/c^{m-1}) c^{m+k} \int_{B(P,r)} v_{m+k} d\rho. \end{aligned} \tag{23}$$

From (22) and (23) it now follows that the remainder term $\int_{B(P,r)} v_{m+k}\Delta^{m+k} w d\rho$ in (16) tends to zero as k tends to ∞. Hence, we have

$$\frac{1}{\Omega_r}\int_{S(P,r)} w d\sigma = \Gamma(n/2)\sum_{\nu=0}^{\infty}(r/2)^{2\nu}\frac{\Delta^\nu w(P)}{\nu!\Gamma(\nu+n/2)}. \tag{24}$$

Similarly,

$$\frac{1}{\Omega_r}\int_{S(P,r)} uw d\sigma = \Gamma(n/2)\sum_{\nu=0}^{\infty}(r/2)^{2\nu}\frac{u(P)\Delta^\nu w(P)}{\nu!\Gamma(\nu+n/2)}. \tag{25}$$

From (24) and (25) we get

$$\int_{S(P,r)} uw d\sigma = u(P)\int_{S(P,r)} w d\sigma, \tag{26}$$

for each $S(P, r)$ which together with its interior $B(P, r)$ lies in R. This means by Theorem 3 of [1] that $u \in S(w, R)$. This completes the proof.

III. As mentioned in the Introduction, we will now give an example to show that in E_n, $n>2$, $S(w, R)$ can be infinite dimensional even if the W. F. w is not an eigenfunction of the Laplace operator.

In E_3 let us consider the following W. F.:

$$w(x_1, x_2, x_3) = e^{x_3} + e^{2x_3}, \quad \text{for all } (x_1, x_2, x_3) \in E_3.$$

Clearly, w is not an eigenfunction of the Laplace operator. We have

$$\Delta^2 w = 5\Delta w - 4w, \qquad w_{x_1} = w_{x_2} = (\Delta w)_{x_1} = (\Delta w)_{x_2} = 0 \tag{27}$$

in E_3. Hence, by Theorem 2, $S(w, E_3)$ is the solution space of the system of elliptic equations:

$$\begin{aligned} w\Delta u + 2\sum_{i=1}^{3} u_{x_i} w_{x_i} &= 0, \\ \Delta w \cdot \Delta u + 2\sum_{i=1}^{3} u_{x_i}(\Delta w)_{x_i} &= 0 \end{aligned} \tag{28}$$

in E_3.

Now, for each positive integer k, let $P_k(x_1, x_2)$ be a harmonic polynomial of degree k in E_2. Then each of the functions

$$f_k(x_1, x_2, x_3) = P_k(x_1, x_2), \qquad k = 1, 2, 3, \cdots, \infty,$$

is defined in E_3 and is harmonic in E_3. Hence,

$$\Delta f_k = 0 = (f_k)_{x_3}$$

in E_3, for all positive integers k. It is clear that each of the functions f_k is a solution of system (24) of equations. Therefore, $f_k \in S(w, E_3)$

for each positive integer k. Since the infinite set of functions $\{f_1, f_2, \cdots, f_k, \cdots\}$ is linearly independent over E_3, it follows that $S(w, E_3)$ is infinite dimensional.

It is also clear that similar types of examples can be constructed in four- and higher-dimensional Euclidean spaces.

Bibliography

1. Anil K. Bose, *Functions satisfying a weighted average property,* Trans. Amer. Math. Soc. **118** (1965), 472–487.

2. ———, *Functions satisfying a weighted average property.* II, Trans. Amer. Math. Soc. **124** (1966), 540–551.

3. R. Courant and D. Hilbert, *Methods of mathematical physics*, Vol. II, Interscience, New York, 1962.

University of Alabama