THE VALUE OF J AT A SAMELSON PRODUCT[1]

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Introduction. At the Seattle Conference, 1963, M. G. Barratt asked the question, "Can anything general be said about $J: \pi_k(SO_n) \to \pi_{k+n}(S^n)$ on Samelson products?" Lashof records [4] that Stasheff pointed out an answer to this question on the basis of results obtained by B. Steer in his Oxford thesis [8]. We offer here another answer, phrased in quite different terms, which is simple in both statement and proof.

Let $p: O_{n+1} \to S^n$ be the usual bundle map with fiber O_n; in [5] we define the "p-product": $\pi_q(O_n) \otimes \pi_s(S^n) \to \pi_{q+s}(S^n)$. The p-product of $\alpha \in \pi_q(O_n)$ with $\beta \in \pi_s(S^n)$ is denoted $[\alpha, \beta]_p$. This product generalizes the J homomorphism: if ι_n denotes the class of the identity map on S^n then $J(\alpha) = [\alpha, \iota_n]_p$ [5, Corollary 5.7].

THEOREM. *Let $\alpha \in \pi_q(O_n)$, $\alpha' \in \pi_{q'}(O_n)$, and let $\langle \alpha, \alpha' \rangle$ denote their Samelson product (see [3]). Then*

$$J\langle \alpha, \alpha' \rangle = [\alpha, J\alpha']_p - (-1)^{qq'}[\alpha', J\alpha]_p.$$

The p-product is defined in terms of the "mixed product":

$$\pi_q(O_n) \otimes \pi_s(O_{n+1}, O_n) \to \pi_{q+s}(O_{n+1}, O_n);$$

the product of $\alpha \in \pi_q(O_n)$ with $\gamma \in \pi_s(O_{n+1}, O_n)$ is denoted $\langle \alpha, \gamma \rangle$, and $[\alpha, \beta]_p = (-1)^q p_* \langle \alpha, p_*^{-1}\beta \rangle$ by definition [5]. Hence our theorem may be restated as

$$\overset{-1}{p_*} J\langle \alpha, \alpha' \rangle = (-1)^q \langle \alpha, \overset{-1}{p_*} J\alpha' \rangle - (-1)^{q'q+q'} \langle \alpha', \overset{-1}{p_*} J\alpha \rangle.$$

Since J may be expressed as a mixed product, this latter statement may readily be translated into a Jacobi identity. We prove the theorem by establishing, for group pairs (G, H) (where H is a subgroup of G), an identity for the mixed product $\pi_q(H) \otimes \pi_s(G, H) \to \pi_{q+s}(G, H)$:

$$(-1)^{qs}\langle\langle \alpha, \alpha' \rangle, \gamma \rangle + (-1)^{q's}\langle\langle \gamma, \alpha \rangle, \alpha' \rangle + (-1)^{qq'}\langle\langle \alpha', \gamma \rangle, \alpha \rangle = 0.$$

This follows from two lemmas, each of some independent interest. (Each loop space below is meant to be of the type introduced by Moore.)

LEMMA 1. *The Jacobi identity holds for the H-pair $(\Omega Y, \Omega B)$ of loop spaces of a pair (Y, B) of spaces.*

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LEMMA 2. *For each group pair* (G, H) *which satisfies the first axiom of countability there exists a classifying pair* (B_G, B_H) *and a weak homotopy equivalence* $f\colon (G, H) \to (\Omega B_G, \Omega B_H)$ *which preserves mixed products.*

(We certainly would expect a stronger form of this lemma to hold.)

The author has appreciated stimulating conversations on this question with Eldon Dyer and Richard Lashof.

Proof of Lemma 1. We introduce a method of "universal examples" for H-constructions which parallels a device of Blakers and Massey [**1**]. Let B be a space (with basepoint), $\partial\colon \pi_{q+1}(B) \cong \pi_q(\Omega B)$ and let $a\colon S^{q+1} \to B$ represent $\partial^{-1}(\alpha)$ for $\alpha \in \pi_q(\Omega B)$. Then a induces an exact H-map $\bar{a}\colon \Omega S^{q+1} \to \Omega B$ defined by $\bar{a}(l) = a \circ l$. Further, if ι_{q+1} is the class of the identity map on S^{q+1} and $\bar{\iota}_q = \partial(\iota_{q+1})$, then $\bar{a}_*(\bar{\iota}_q) = \alpha$. A modification of the above for a pair (Y, B) uses $\partial\colon \pi_{s+1}(Y, B) \cong \pi_s(\Omega Y, \Omega B)$ to define from $\gamma \in \pi_s(\Omega Y, \Omega B)$ a map $c\colon (I^{s+1}, S^s) \to (Y, B)$ and, in turn, an exact H-map $\bar{c}\colon (\Omega I^{s+1}, \Omega S^s) \to (\Omega Y, \Omega B)$. If κ_{s+1} is the class of the identity on (I^{s+1}, S^s) and $\bar{\kappa}_s = \partial(\kappa_{s+1}) \in \pi_s(\Omega I^{s+1}, \Omega S^s)$ then $\bar{c}_*(\bar{\kappa}_s) = \gamma$. Again, if α and γ together define a map from $(S^{q+1} \vee I^{s+1}, S^{q+1} \vee S^s)$ into (Y, B) and thus an exact H-map $g\colon [\Omega(S^{q+1} \vee I^{s+1}), \Omega(S^{q+1} \vee S^s)] \to (Y, B)$, and if $\bar{\iota}_q$ and $\bar{\kappa}_s$ denote their own images under appropriate inclusions into $\Omega(S^{q+1} \vee S^s)$ and $[\Omega(S^{q+1} \vee I^{s+1}), \Omega(S^{q+1} \vee S^s)]$, then the naturality of the mixed product implies

$$g_*\langle \bar{\iota}_q, \bar{\kappa}_s \rangle = \langle g_*\bar{\iota}_q, g_*\bar{\kappa}_s \rangle = \langle \alpha, \gamma \rangle.$$

Yet another such construction uses representatives of $\alpha \in \pi_q(\Omega B)$, $\alpha' \in \pi_{q'}(\Omega B)$ and $\gamma \in \pi_s(\Omega Y, \Omega B)$ to construct an exact H-map

$$h\colon [\Omega(S^{q+1} \vee S^{q'+1} \vee I^{s+1}), \Omega(S^{q+1} \vee S^{q'+1} \vee S^s)] \to (\Omega Y, \Omega B)$$

with the property that $h_*\langle\langle \bar{\iota}_q, \bar{\iota}_{q'} \rangle, \bar{\kappa}_s \rangle = \langle\langle \alpha, \alpha' \rangle, \gamma \rangle$, etc. Now observe that in the homotopy sequence

$$\begin{aligned} \cdots \to \pi_n[\Omega(S^{q+1} \vee S^{q'+1} \vee I^{s+1}), \Omega(S^{q+1} \vee S^{q'+1} \vee S^s)] \\ \xrightarrow{\partial} \pi_{n-1}[\Omega(S^{q+1} \vee S^{q'+1} \vee S^s)] \\ \xrightarrow{i} \pi_{n-1}[\Omega(S^{q+1} \vee S^{q'+1} \vee I^{s+1})] \to \cdots \end{aligned}$$

i is an epimorphism in all dimensions, so that ∂ is a monomorphism. But the relations $\partial\langle \alpha, \gamma \rangle = \langle \alpha, \partial\gamma \rangle$ and $\partial\langle \gamma, \alpha \rangle = (-1)^q \langle \partial\gamma, \alpha \rangle$ between the mixed products in the domain of ∂ and the absolute (Samelson) products in the range carry a "Jacobi sum" in the domain to a Jacobi sum in the range. Since the Jacobi relation holds for the absolute product [**3**], it must hold in the domain of ∂. It is now clear that if the morphism h_* is defined by α, α' and γ then

$$\begin{aligned} h_*((-1)^{qs}\langle\langle\bar{\iota}_q, \bar{\iota}_{q'}\rangle, \bar{\kappa}_s\rangle + (-1)^{sq'}\langle\langle\bar{\kappa}_s, \bar{\iota}_q\rangle, \bar{\iota}_{q'}\rangle + (-1)^{q'q}\langle\langle\bar{\iota}_{q'}, \bar{\kappa}_s\rangle, \bar{\iota}_q\rangle) \\ = (-1)^{qs}\langle\langle\alpha, \alpha'\rangle, \gamma\rangle + (-1)^{sq'}\langle\langle\gamma, \alpha\rangle, \alpha'\rangle + (-1)^{q'q}\langle\langle\alpha', \gamma\rangle, \alpha\rangle \\ = 0 \in \pi_{q+q'+s}(\Omega Y, \Omega B). \end{aligned}$$

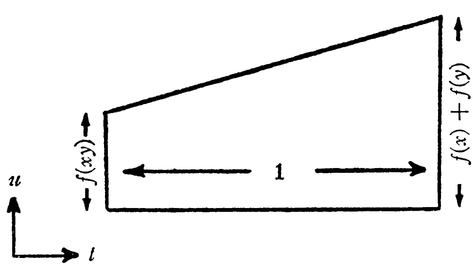
Proof of Lemma 2. Let (G, H) be a group pair; Milnor [6] gives a construction of a universal bundle E_G with base space B_G and fiber G. Examination of this construction shows that the corresponding universal bundle $E_H \rightarrow B_H$ with fiber H is a subbundle of $E_G \rightarrow B_G$; that is, $(G, H) \rightarrow (E_G, E_H) \rightarrow (B_G, B_H)$ is a fibering of pairs. An explicit recipe for a contraction of E_G is offered by Dold [2]; his function moves E_H through E_H to a point, and so is a contraction of the pair (E_G, E_H). Samelson [7] describes an H-map from G to the usual loop space of B_G which is a weak homotopy equivalence; his map may be constructed from a given contraction k of E_G and it thus defines a map of the pair (G, H) into the pair of usual loop spaces of B_G and B_H. This pair map is a weak homotopy equivalence, by the five lemma.

We now alter Samelson's map to define a weak homotopy equivalence from (G, H) into the pair $(\Omega B_G, \Omega B_H)$ of Moore-loop spaces. Let k: $E_G \times I \rightarrow E_G$ be the contraction of E_G to $e \in G$ described by Dold, and let f be a nonnegative real valued function on G with $f^{-1}(0) = e$ (the existence of such an f is easily shown for first countable groups). For each $x \in G$ define $\phi(x) \in \Omega B_G$ to be the loop $\phi(x)$: $[0, f(x)] \rightarrow B_G$ whose values are, for $x \neq e$,

$$\phi(x)(u) = p \circ k[x, u/f(x)].$$

(Here p: $(E_G, E_H) \rightarrow (B_G, B_H)$ is the bundle projection.) Since ϕ is clearly homotopic to the map defined by Samelson when the latter is regarded as having range ΩB_G, ϕ is a weak homotopy equivalence, and $\phi(H) \subset \Omega B_H$.

Next we define a homotopy $\mathcal{H}$: $G \times G \times I \rightarrow \Omega B_G$ which shows ϕ to be a strong H-map (that is, $\mathcal{H}(x, e, t) = \mathcal{H}(e, x, t)$ for all t) of pairs.



The existence of $\mathcal{H}$ implies that ϕ preserves mixed products [**5**, Section 4]. For simplicity of notation, we identify (G, H) with the distinguished fiber of (E_G, E_H) and denote the right action of (G, H) on (E_G, E_H) by juxtaposition; e is the identity of G. For each $(x, y) \in G\times G$, let $h_{x,y}$ be the E_G-valued function defined on a $(t-u)$-trapezoid as follows: on the bottom edge $h_{x,y}(t, 0)=xy$, on the left edge $h_{x,y}(0, u)=k[xy, u/f(xy)]$, on the top edge $h_{x,y}(t, u)=e$, and along the right edge

$$\begin{aligned} h_{x,y}(1, u) &= k[x, u/f(x)]y && \text{if } 0 \leqq u \leqq f(t), \\ &= k[y, (u-f(x))/f(y)] && \text{if } f(x) \leqq u \leqq f(x)+f(y). \end{aligned}$$

Now let $h_{x,y}$ be defined at the center of the trapezoid by

$$h_{x,y}\left(\frac{1}{2}, \frac{f(xy)+f(x)+f(y)}{2}\right) = e$$

and extend $h_{x,y}$ to the whole figure by letting the contraction k act in the obvious linear fashion along each ray from center to boundary. The only degenerate case is $x=y=e$; there $h_{e,e}=e$. Let p be the projection of the bundle (E_G, E_H) and define $\mathcal{H}(x, y, t)(u)=p\circ h_{x,y}(t, u)$; since the right action of G on E_G has trivial projection, each value of $\mathcal{H}$ is clearly a Moore loop in B_G. Further, $\mathcal{H}(x, y, 0)=\phi(x, y)$, $\mathcal{H}(x, y, 1)=\phi(x)\phi(y)$, and $\mathcal{H}(x, e, t)=\mathcal{H}(e, x, t)$ for all x and t; hence ϕ is a strong H-map.

References

1. A. L. Blakers and W. S. Massey, *Products in homotopy theory*, Ann. of Math. **58** (1953), 295–324.

2. A. Dold, *Partitions of unity in the theory of fibrations*, Ann. of Math. **78** (1963), 223–255; p. 251.

3. I. M. James, *On H-spaces and their homotopy groups*, Quart. J. Math. Oxford Ser. (2) **11** (1960), 161–179; p. 170.

4. R. Lashof, Editor, *Problems in differential and algebraic topology*, Seattle Conference, 1963, Ann. of Math. **81** (1965), 565–591; p. 588.

5. G. McCarty, *Products between homotopy groups and the J-morphism*, Quart. J. Math. Oxford Ser. (2) **15** (1964), 362–370.

6. J. Milnor, *Construction of universal bundles*. II, Ann. of Math. **63** (1956), 430–436.

7. H. Samelson, *Groups and spaces of loops*, Comment. Math. Helv. **28** (1954), 278–287.

8. B. Steer, *Extensions of mappings into H-spaces*, Proc. London Math. Soc. (3) **13** (1963), 219–272; Theorem 5.76.

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