THE EILENBERG-MOORE, ROTHENBERG-STEENROD SPECTRAL SEQUENCE FOR $K$ THEORY

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Let $G$ be a topological group and let $G \to E \to B_G$ be the universal fibration for $G$ so that $B_G$ is the classifying space for $G$. Eilenberg and Moore have developed two spectral sequences which deal with this situation [1], [2], [3]. The spectral sequence of Type I has $E_2 = \text{Cotor}^{H^*(G)}(K, K) \Rightarrow H^*[B_G]$. The Type II spectral sequence has $E_2 = \text{Tor}^{H^*[B_G]}(K, K) \Rightarrow H^*[G]$. $K$ is the ground ring. These tools give a very simple proof of

**Theorem $H^*$.** $H^*[G]$ is an exterior algebra over $K$ iff $H^*[B_G]$ is a polynomial ring over $K$.

The Eilenberg-Moore construction of these spectral sequences was almost entirely algebraic [1].

Rothenberg and Steenrod have studied the Type I spectral sequence and have produced a geometric construction which gives rise to it, [4]. Their approach has the advantage that being of geometric origin it applies to other cohomology theories such as the Atiyah-Hirzebruch complex $K$ theory [5]. In particular, if $G$ is a Lie group and $K^*[G]$ is an exterior algebra over $Z$, then one has a spectral sequence $E_2 = \text{Cotor}^{K^*[G]}[Z, Z] \Rightarrow K^*[B_G]$. The spectral sequence collapses and hence gives an easy proof of

**Theorem $K^*$.** If $G$ is a Lie group for which $K^*[G]$ is an exterior algebra over $Z$, then $K^*[B_G]$, the completed representation ring of $G$, is a power series ring.

The natural question arises as to whether the Type II spectral sequence exists in $K$ theory. The purpose of this note is to show that it does not by showing that the converse of Theorem $K^*$ is false.

**Counterexample.** There does not exist a Moore-Eilenberg spectral sequence of Type II in $K$ theory for the fibration $SO(3) \to E \to B_{SO(3)}$.

**Proof.** $K^*[SO(3)] = Z \oplus Z \oplus Z_2$. $K^*[B_{SO(3)}]$ is the completed representation ring of $SO(3)$ which is a power series ring $Z[[\rho]]$. Suppose that such a spectral sequence $E_2 = \text{Tor}^{K^*[B_{SO(3)}]}[Z, Z] \Rightarrow K^*[SO(3)]$ exists. An easy computation shows that the $E_2$ term $\text{Tor}^{2[[\rho]]}[Z, Z]$ is an exterior algebra $E[S^{-1}\rho] = Z \oplus Z S^{-1}\rho$ generated by the desuspension of $\rho$. Since the $E_2$ term $E[S^{-1}\rho]$ is already “smaller than” $K^*[SO(3)]$ we have a contradiction.

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REFERENCES


