THE EILENBERG-MOORE, ROTHENBERG-STEENROD SPECTRAL SEQUENCE FOR K THEORY

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Let G be a topological group and let $G \rightarrow E \rightarrow B_G$ be the universal fibration for G so that B_G is the classifying space for G. Eilenberg and Moore have developed two spectral sequences which deal with this situation [1], [2], [3]. The spectral sequence of Type I has $E_2 = \mathrm{Cotor}^{H^*(G)}(K, K) \Rightarrow H^*[B_G]$. The Type II spectral sequence has $E_2 = \mathrm{Tor}^{H^*[B_G]}(K, K) \Rightarrow H^*[G]$. K is the ground ring. These tools give a very simple proof of

THEOREM H^*. *$H^*[G]$ is an exterior algebra over K iff $H^*[B_G]$ is a polynomial ring over K.*

The Eilenberg-Moore construction of these spectral sequences was almost entirely algebraic [1].

Rothenberg and Steenrod have studied the Type I spectral sequence and have produced a geometric construction which gives rise to it, [4]. Their approach has the advantage that being of geometric origin it applies to other cohomology theories such as the Atiyah-Hirzebruch complex K theory [5]. In particular, if G is a Lie group and $K^*[G]$ is an exterior algebra over Z, then one has a spectral sequence $E_2 = \mathrm{Cotor}^{K^*[G]}[Z, Z] \Rightarrow K^*[B_G]$. The spectral sequence collapses and hence gives an easy proof of

THEOREM K^*. *If G is a Lie group for which $K^*[G]$ is an exterior algebra over Z, then $K^*[B_G]$, the completed representation ring of G, is a power series ring.*

The natural question arises as to whether the Type II spectral sequence exists in K theory. The purpose of this note is to show that it does not by showing that the converse of Theorem K^* is false.

COUNTEREXAMPLE. *There does not exist a Moore-Eilenberg spectral sequence of Type* II *in K theory for the fibration $SO(3) \rightarrow E \rightarrow B_{SO(3)}$.*

PROOF. $K^*[SO(3)] = Z \oplus Z \oplus Z_2$. $K^*[B_{SO(3)}]$ is the completed representation ring of $SO(3)$ which is a power series ring $Z[[\rho]]$. Suppose that such a spectral sequence $E_2 = \mathrm{Tor}^{K^*[B_{SO(3)}]}[Z, Z] \Rightarrow K^*[SO(3)]$ exists. An easy computation shows that the E_2 term $\mathrm{Tor}^{Z[[\rho]]}[Z, Z]$ is an exterior algebra $E[S^{-1}\rho] = Z \oplus ZS^{-1}\rho$ generated by the desuspension of ρ. Since the E_2 term $E[S^{-1}\rho]$ is already "smaller than" $K^*[SO(3)]$ we have a contradiction.

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