# ON THREE BASIC RESULTS IN THE THEORY OF STATIONARY POINT PROCESSES[1]

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**Summary.** A simple, unified treatment is given for three basic results in the theory of stationary streams of events (point processes). These results are the basic theorem of Khintchine concerning the existence of a stream intensity, Korolyuk's theorem, and a fundamental lemma of Dobrushin giving sufficient conditions for a stationary stream to be *orderly* in the sense of Khintchine [**2**].

1. **Introduction.** Consider a stationary stream of events and let $N(s, t)$ denote the number of events occurring in the interval $(s, t)$. We note here that it is possible to define a basic probability space in a number of ways so that $N(s, t)$ is a well defined random variable (cf. Ryll Nardzewski [**4**], Matthes [**3**], Cramér and Leadbetter [**1**]). We shall not be concerned with this basic structure except through the properties of random variables of the form $N(s, t)$. We further note that the *stationarity* of the stream is to be taken to mean that the joint distribution of the random variables

$$N(s_1 + \tau, t_1 + \tau), \cdots, N(s_k + \tau, t_k + \tau)$$

is independent of $\tau$ for any fixed $k$ and choice of $(s_i, t_i)$, $i = 1, \cdots, k$. *Khintchine's existence theorem* asserts that for such a stationary stream of events there exists a nonnegative constant $\lambda \leqq \infty$ such that $P\{N(0, t) \geqq 1\} \sim \lambda t$ as $t \to 0$.

Following Khintchine [**2**], a stationary stream of events will be called *orderly* if $P\{N(0, t) > 1\} = o(t)$ as $t \to 0$. *Korolyuk's theorem* [**2**] states that, for a stationary, orderly stream we have $\lambda = \mathcal{E}N(0, 1)$.

In general multiple events may occur—that is it is possible to have more than one event at the same time point. It is easy to see that this is not possible if the stream is orderly. Conversely *Dobrushin's lemma* [**5**] states that if multiple events cannot occur and if $\mathcal{E}N(0, 1) < \infty$, then the stationary stream is orderly.

Separate proofs of the three theorems mentioned above are available (cf. [**2**], [**5**], [**1**]). In the next section we shall give very short proofs for all three theorems by means of a simple technique.

2. **The basic theorems.** It is clear that we may modify a given stationary stream of events to form a new stationary stream by consider-

Received by the editors October 18, 1966.

[1] Research sponsored by the Office of Naval Research, Contract Nonr 855(09).



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ing multiple events as single events in the modified stream. Write $N^*(s, t)$ for the number of such (modified) events in $(s, t)$; that is the number of original events in $(s, t)$ counted without regard for their possible multiplicities. It follows at once that $N^*(s, t) \geqq 1$ if and only if $N(s, t) \geqq 1$.

We first prove the following result, which contains both the theorem of Khintchine and Korolyuk's theorem.

THEOREM 1. *Consider a stationary stream of events, and with the above notation, write* $\mu^* = \mathcal{E}N^*(0, 1) \leqq \infty$. *Then*

$$P\{N(0, t) \geqq 1\} \sim \mu^* t \quad as \quad t \to 0. \tag{1}$$

PROOF. For a given integer $n$ write $\chi_{in} = 1$ if $N(i/n, (i+1)/n) \geqq 1$ and $\chi_{in} = 0$ otherwise, $i = 0, 1, \cdots, n-1$. Let $N_n = \sum_{i=0}^{n-1} \chi_{in}$. Then it is easy to see that $N_n \to N^*(0, 1)$ with probability one, as $n \to \infty$. If $\mu^* < \infty$ it follows at once by dominated convergence that $\mathcal{E}N_n \to \mathcal{E}N^*(0, 1)$ and hence from stationarity that

$$n\mathcal{E}\chi_{0n} = nP\{N(0, n^{-1}) \geqq 1\} \to \mu^* \quad \text{as } n \to \infty.$$

The required result then follows from the obvious inequalities

$$\begin{aligned} \frac{t^{-1}}{([t^{-1}]+1)} \frac{P\{N(0, [([t^{-1}]+1)^{-1}) \geqq 1\}}{([t^{-1}]+1)^{-1}} &\leqq \frac{P\{N(0, t) \geqq 1\}}{t} \\ &\leqq \frac{P\{N(0, [t^{-1}]^{-1}) \geqq 1\}}{[t^{-1}]^{-1}} \frac{t^{-1}}{[t^{-1}]} \end{aligned}$$

(where $[x]$ denotes the integer part of $x$), since the outside terms each tend to $\mu^*$ as $t \to 0$.

If $\mu^* = \infty$ an application of Fatou's lemma in place of dominated convergence easily yields (1).

This result shows at once that a stationary stream has an intensity $\lambda = \mu^*$, i.e. the theorem of Khintchine follows. Further, if multiple events have probability zero, $\mu^* = \mathcal{E}N(0, 1)$ and hence $\lambda = \mathcal{E}N(0, 1)$. This is a slightly sharper form of Korolyuk's theorem since orderliness of the stream implies, as noted, that multiple events have probability zero.

Finally, we prove Dobrushin's lemma by a slight modification of the above technique.

THEOREM 2. *Suppose the stream of events considered is stationary, that the probability of the occurrence of multiple events is zero, and that* $\mathcal{E}N(0, 1) < \infty$. *Then the stream is also orderly.*

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PROOF. Write $\chi'_{in}=1$ if $N(i/n,\ (i+1)/n)>1$, $\chi'_{in}=0$ otherwise, $i=0, 1, \cdots, n-1$. Let $N_n' = \sum_{i=0}^{n-1} \chi'_{in}$. Then $N_n'$ is the number of intervals $(i/n,\ (i+1)/n)$ containing at least two events and since multiple events have probability zero, $N_n' \to 0$ with probability one. But $N_n' \leqq N(0,\ 1)$ and hence by dominated convergence, $\mathcal{E}N_n' \to 0$ as $n \to \infty$. By stationarity we thus have $nP\{N(0,\ n^{-1})>1\} \to 0$ as $n \to \infty$. Hence

$$\frac{P\{N(0, t) > 1\}}{t} \leqq \frac{P\{N(0, [t^{-1}]^{-1}) > 1\}}{[t^{-1}]^{-1}} \frac{t^{-1}}{[t^{-1}]} \to 0 \quad \text{as} \quad t \to 0.$$

## REFERENCES

1. Harald Cramér and M. R. Leadbetter, *Stationary and related stochastic processes*, Wiley, New York, 1966.

2. Y. A. Khintchine, *Mathematical methods in the theory of queueing*, Griffin, London, 1960.

3. K. Matthes, *Stationäre Zufällige Punktfolgen*. I, Jber. Deutsch. Math.-Verein. **66** (1963), 66–79.

4. C. Ryll Nardzewski, *Remarks on processes of calls*, Proc. Fourth Berkeley Sympos. Math. Stat. & Prob., Vol. 2, 1961, pp. 455–466.

5. V. A. Volkonski, *An ergodic theorem on the distribution of the duration of fades*, Teor. Verojatnost. i Primenen. **5** (1960), 360–361.

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