A PROPERTY OF FREE BOOLEAN ALGEBRAS

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Consider the following properties of a Boolean algebra A:

P_1: Every set of pairwise disjoint elements of A is countable.

P_2: Every chain in A is countable.

For arbitrary Boolean algebras neither of these properties implies the other. The algebra of all sets of integers satisfies P_1 but not P_2, while the algebra of all finite or cofinite sets of real numbers satisfies P_2 but not P_1. It is well known that P_1 holds in any free Boolean algebra. However it is not generally realized that P_2 also holds in free Boolean algebras. In fact this statement has not explicitly appeared in the literature, although it is a consequence of a theorem in topology due to N. A. Šanin [**1**, Theorem 50]. The following is a simple direct proof of the statement.

THEOREM. *Every chain in a free Boolean algebra is countable.*

PROOF. Let T be a set of cardinality α and let M be the set of all functions on T with values 0 or 1. For each $t \in T$ let $D_t = \{f \in M: f(t) = 1\}$. The algebra F generated by the sets D_t is the free Boolean algebra with α free generators. Now suppose C is an uncountable chain in F. We may assume $0 \notin C$. For each member x of C there is a finite subset A of T such that x is in the subalgebra generated by $\{D_t: t \in A\}$. Therefore there exist distinct subsets $A_1, \cdots, A_r$ of A such that

$$x = \bigcup_{i=1}^{r} \left[\bigcap_{t \in A_i} D_t \cap \bigcap_{t \in A - A_i} \overline{D}_t \right]. \tag{1}$$

Thus to each member of C we can associate a pair (n, r) of positive integers, where n is the cardinality of A, and r is the number of subsets A_i. Since C is uncountable there will certainly be distinct members x and y of C associated with the same pair (n, r). Then there exist subsets A and B of T with cardinality n, distinct subsets $A_1, \cdots, A_r$ of A, and distinct subsets $B_1, \cdots, B_r$ of B such that (1) holds, and

$$y = \bigcup_{i=1}^{r} \left[\bigcap_{t \in B_i} D_t \cap \bigcap_{t \in B - B_i} \overline{D}_t \right]. \tag{2}$$

Since C is a chain we may assume $x \subset y$. We will reach a contradiction by showing $x = y$.

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If f is any member of M and $S=\{t: f(t)=1\}$, then $f\in x$ if and only if $S\cap A=A_i$ for some i. Since $x\subset y$, it follows that for any subset S of T,

(3) if $S\cap A = A_i$ for some i, then $S\cap B = B_j$ for some j.

Let $C_1, \cdots, C_p$ be the distinct members of $\{A_i\cap B: 1\leqq i\leqq r\}$ and q be the cardinality of $A-B$, which is also equal to the cardinality of $B-A$. If K is any subset of $B-A$, then by (3) with $S=A_i\cup K$, we see that for each k, $1\leqq k\leqq p$, there is a j such that $K\cup C_k=B_j$. Therefore $r\geqq p\cdot 2^q$. However we also have $r\leqq p\cdot 2^q$, since each A_i is of the form $C_k\cup K$ for some k, and some subset K of $A-B$. Thus $r=p\cdot 2^q$, and hence

(4) $\{B_i: 1\leqq i\leqq r\} = \{C_k\cup K: 1\leqq k\leqq p \text{ and } K\subseteq B-A\}$

and

(5) $\{A_i: 1\leqq i\leqq r\} = \{C_k\cup K: 1\leqq k\leqq p \text{ and } K\subseteq A-B\}$.

Let $D=A\cap B$. Then by (2) and (4),

$$\begin{aligned} y &= \bigcup_{k=1}^{p}\bigcup_{K\subseteq B-A}\left[\bigcap_{t\in C_k\cup K} D_t\cap\bigcap_{t\in B-(C_k\cup K)}\overline{D}_t\right] \\ &= \bigcup_{k=1}^{p}\left[\bigcap_{t\in C_k} D_t\cap\bigcap_{t\in D-C_k}\overline{D}_t\right]\cap\bigcup_{K\subseteq B-A}\left[\bigcap_{t\in K} D_t\cap\bigcap_{t\in(B-A)-K}\overline{D}_t\right] \\ &= \bigcup_{k=1}^{p}\left[\bigcap_{t\in C_k} D_t\cap\bigcap_{t\in D-C_k}\overline{D}_t\right]. \end{aligned}$$

Similarly

$$x = \bigcup_{k=1}^{p}\left[\bigcap_{t\in C_k} D_t\cap\bigcap_{t\in D-C_k}\overline{D}_t\right]$$

and so $x=y$.

Reference

1. N. A. Šanin, *O proizvedenii topologičlskih prostranstv*, Trudy Mat. Inst. Steklov. **24** (1948), 112 pp.

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