THE EQUATION $(I-S)g=f$ FOR SHIFT OPERATORS IN HILBERT SPACE

RICHARD ROCHBERG

1. **Introduction.** The purpose of this paper is to establish some conditions on f for the equation $f=g-Sg$ to be solvable for g in a Hilbert space, with S a shift operator on that space.

The following theorem was stated by R. Fortet [**4**] and proved by M. Kac [**3**].

THEOREM 1. *If f is a function periodic of period* 1, *satisfying in* (0, 1) *the Hölder condition of order α for some $\alpha>\frac{1}{2}$, and further $\int_0^1 f(t)dt=0$, then the condition*

$$\lim_{n\to\infty}\frac{1}{n}\int_0^1\left|\sum_{i=0}^{n}f(2^it)\right|^2 dt = 0 \tag{1}$$

is necessary and sufficient for the existence of a function g in $L^2(0, 1)$ such that

$$f(t) = g(t) - g(2t) \quad a.e. \tag{2}$$

Kac's method is to find formally the Fourier coefficients of g in (2), use the Hölder condition on f and the condition (1) to estimate the size of these coefficients and show that they are, in fact, the Fourier coefficients of a function in $L^2(0, 1)$.

Z. Ciesielski has since extended the theorem to the case $\alpha>0$ [**1**]. The basic difference between Ciesielski's approach and Kac's is the use of Fourier-Haar coefficients in place of Fourier coefficients.

It will be the purpose of this paper to prove a theorem about inverting certain operators on Hilbert space. Theorem 1 will then be shown to be an immediate consequence of our more general theorem. First however, we introduce some definitions and established results.

2. **Shift operators.** Let H be a Hilbert space. We define a *shift operator* on H as a linear isometry on H which is unitary on no nontrivial subspace.

As an example of a shift operator let L be the subspace of $L^2(0, 1)$ of functions whose zeroth Fourier coefficient vanishes, S the linear operator on L defined by $(Sf)(t)=f(2t)$. In this context, Theorem 1 becomes a set of conditions on f for the existence of an element

Received by the editors November 19, 1966.

$g=(I-S)^{-1}f$ in the Hilbert space L with S a specific shift operator on L.

We now introduce some notation which will remain fixed for the rest of the paper. S will be a shift operator on the Hilbert space H. K will be the orthogonal complement of the range of S. For any non-negative integer j let P_j be the projection from H into the subspace $S^j(K)$, $P^j=\sum_{i=0}^{j} P_i$, and $R_j=I-P^j$ (I, the identity on H). To shorten formulas, for f in H, and only for f, we shall write $P_j(f)=f_j$, $P^j(f)=f^j$, and $R_j(f)=R_j$. The following standard result will be used (Halmos [2]).

THEOREM 2. *If S is a shift operator on the Hilbert space H, and K is the orthogonal complement of the range of S, then $H=K\oplus S(K)\oplus S^2(K)\oplus\cdots$, and $S^i(K)\perp S^j(K)$ for $i\neq j$.*

A large number of simple facts such as $f=\lim f^j$ and $(f^i, S^{i+1}f)=0$ follow immediately from the decomposition of H and our notation. Such results will be used throughout the paper, often without mention.

3. **The main thoerem.**

We can now state the main result of this paper.

THEOREM 3. *With the above notation, let S be a shift operator on the Hilbert space H and f an element of H. If for some $\beta>0$*

$$\|f_j\| = \theta(2^{-\beta j}) \tag{3}$$

then a necessary and sufficient condition for the existence of a g in H with $(I-S)g=f$ is

$$\lim_{n\to\infty}\frac{1}{n}\left\|\sum_{i=0}^{n}S^i f\right\|^2=0. \tag{4}$$

The formal expansion $(I-S)^{-1}=\sum_{i=0}^{\infty}S^i$ suggests that $g=\sum_{i=0}^{\infty}S^i(\sum_{j=0}^{\infty}f_j)$. Unfortunately this series does not converge. The proof of Theorem 3 involves rearranging this series to $g=\sum_{r=0}^{\infty}(\sum_{k=0}^{r}S^k f_{r-k})$ and showing that the rearranged series converges to the desired element. An alternate proof of Theorem 3 is outlined in the next section. Before proving the theorem we will prove four lemmas. The first two were proved by Kac [3] for the specific case of Theorem 1.

LEMMA 3.1. *If f satisfies (3) then $(f, S^k f)=O(2^{-\beta k})$.*

PROOF. We know that $f=\sum_{i=0}^{\infty}f_i$ and that $(f_i, S^k f_j)=0$ if $k+j\neq i$, hence

$$\begin{aligned} |(f, S^k f)| &= \left| \sum_{i=k}^{\infty} (f_i, S^k f_{i-k}) \right| \leqq \sum_{i=k}^{\infty} \|f_i\| \, \|S^k f_{i-k}\| \\ &= \sum_{i=k}^{\infty} \|f_i\| \, \|f_{i-k}\| \leqq \left(\sum_{i=k}^{\infty} \|f_{i-k}\|^2 \right)^{1/2} \left(\sum_{i=k}^{\infty} \|f_i\|^2 \right)^{1/2} \\ &\leqq \|f\| \left(\sum_{i=k}^{\infty} O(2^{-\beta i})^2 \right)^{1/2} = O(2^{-\beta k}) \end{aligned}$$

as was to be shown.

LEMMA 3.2. *If f satisfies* (3) *then*

$$\lim_{n\to\infty} \frac{1}{n} \left\| \sum_{k=0}^{n} S^k f \right\|^2 = \|f\|^2 + 2 \operatorname{Re} \sum_{k=1}^{\infty} (f, S^k f).$$

PROOF.

$$\begin{aligned} \frac{1}{n} \left\| \sum_{k=0}^{n} S^k f \right\|^2 &= \frac{1}{n} \left(\sum_{i=0}^{n} (S^i f, S^i f) + 2 \operatorname{Re} \sum_{0 \leq i < j \leq n} (S^i f, S^j f) \right) \\ &= \|f\|^2 + \frac{2}{n} \operatorname{Re} \sum_{0 \leq i < j \leq n} (f, S^{j-i} f) \\ &= \|f\|^2 + \frac{2}{n} \operatorname{Re} \left(\sum_{r=1}^{n} (n - r + 1)(f, S^r f) \right) \\ &= \|f\|^2 + 2 \operatorname{Re} \sum_{r=1}^{n} (f, S^r f) - \frac{2}{n} \operatorname{Re} \sum_{r=1}^{n} (r - 1)(f, S^r f) \end{aligned}$$

but by the estimate of Lemma 3.1, both sums in the previous line converge as n goes to infinity. Hence $2/n$ times the last sum goes to zero. Therefore, taking the limit as n goes to infinity at the beginning and end of the above equation we have the desired result.

LEMMA 3.3: *If f satisfies* (3) *then* $\|R_j\| = \theta(2^{-\beta j})$.

PROOF: By the definition of R_j, (3), and the subspace decomposition of H we have

$$\|R_j\| = \left(\sum_{i=j+1}^{\infty} \|f_i\|^2 \right)^{1/2} = \left(\sum_{i=j+1}^{\infty} O(2^{-\beta i})^2 \right)^{1/2} = \theta(2^{-\beta j}).$$

LEMMA 3.4: *For any f in H*

$$\left\| \sum_{i=0}^{r} S^i f_{r-i} \right\|^2 = \lim_{n\to\infty} \frac{1}{n} \left\| \sum_{i=0}^{n} S^i f^r \right\|^2.$$

PROOF. The identity

$$\|f^r\|^2 + 2\ \mathrm{Re} \sum_{k=1}^{\infty} (f^r, S^k f^r) = \left\| \sum_{i=0}^{r} S^i f_{r-i} \right\|^2$$

can be verified by direct computation. This identity and Lemma 3.2 (applied to f^r) together imply the desired result.

PROOF OF THEOREM 3. Throughout the proof, condition (3) will be assumed without being explicitly mentioned. The implication one way, when g is known to exist, is trivial. For the implication in the other direction define $g_r = \sum_{k=0}^{r} S^k f_{r-k}$. Clearly g_r is in $S^r(K)$. We shall show that $\sum_{k=0}^{\infty} \|g_k\|^2 < \infty$. Hence $g = \sum_{k=0}^{\infty} g_k$ is in H. From the definition, it is clear that g has the desired property.

By the parallelogram law we have the following identity:

$$\frac{2}{n} \left\| \sum_{i=0}^{n} S^i f^r \right\|^2 = \frac{1}{n} \left\| \sum_{i=0}^{n} S^i f \right\|^2 + \frac{1}{n} \left\| \sum_{i=0}^{n} S^i f^r - \sum_{i=0}^{n} S^i R_r \right\|^2 - \frac{2}{n} \left\| \sum_{i=0}^{n} S^i R_r \right\|^2 .$$

Letting n go to infinity in the above and using Lemma 3.4 and condition (4) of the theorem this becomes

$$2\|g_r\|^2 = \lim_{n\to\infty} \frac{1}{n} \left\| \sum_{i=0}^{n} S^i (f^r - R_r) \right\|^2 - 2 \lim_{n\to\infty} \frac{1}{n} \left\| \sum_{i=0}^{n} S^i R_r \right|^2 .$$

Lemma 3.2 applies to both $(f^r - R_r)$ and R_r, hence

$$\begin{aligned} 2\|g_r\|^2 &= \|f^r - R_r\|^2 - 2\|R_r\|^2 \\ &\quad + 2\ \mathrm{Re} \sum_{k=1}^{\infty} ((f^r - R_r, S^k(f^r - R_r)) - 2(R_r, S^k R_r)) \\ &= \|f^r\|^2 - \|R_r\|^2 + 2\ \mathrm{Re} \sum_{k=1}^{\infty} [(f^r, S^k f^r) - (f^r, S^k R_r) - (R_r, S^k f^r) \\ &\qquad + (R_r, S^k R_r) - 2(R_r, S^k R_r)] \\ &= \|f^r\|^2 - \|R_r\|^2 + 2\ \mathrm{Re} \sum_{k=1}^{\infty} (f^r, S^k f^r) - 2\ \mathrm{Re} \sum_{k=1}^{\infty} (R_r, S^k f). \end{aligned}$$

But by Lemmas 3.2 and 3.4

$$\begin{aligned} \|f^r\|^2 + 2\ \mathrm{Re} \sum_{k=1}^{\infty} (f^r, S^k f^r) &= \lim_{n\to\infty} \frac{1}{n} \left\| \sum_{k=0}^{n} S^k f^r \right\|^2 \\ &= \left\| \sum_{i=0}^{r} S^i f_{r-i} \right\|^2 = \|g_r\|^2. \end{aligned}$$

Substituting this equality in the previous equation we have

$$\|g_r\|^2 = -\|R_r\|^2 - 2 \operatorname{Re} \sum_{k=1}^{\infty} (R_r, S^k f).$$

we want to show $\sum_{r=0}^{\infty}\|g_r\|^2 < \infty$. By direct application of Lemma 3.3, $\sum_{r=0}^{\infty}\|R_r\|^2 < \infty$. Therefore we need only show

$$\sum_{r=0}^{\infty} \left| \sum_{k=1}^{\infty} (R_r, S^k f) \right| < \infty. \tag{6}$$

Using Lemma 3.3 and the fact that $(R_k - R_r, S^k f) = 0$ if $k > r$, we see

$$\begin{aligned} \left| \sum_{k=1}^{\infty} (R_r, S^k f) \right| &= \left| \sum_{k=1}^{r} (R_r, S^k f) + \sum_{k=r+1}^{\infty} (R_r, S^k f) \right| \\ &\leqq \sum_{k=1}^{r} |(R_r, S^k f)| + \sum_{k=r+1}^{\infty} |(R_k, S^k f)| \\ &\leqq \sum_{k=1}^{r} \|R_r\| \, \|S^k f\| + \sum_{k=r+1}^{\infty} \|R_k\| \, \|S^k f\| \\ &= \|f\| \left(r\|R_r\| + \sum_{k=r+1}^{\infty} \|R_k\| \right) \\ &= \|f\| (rO(2^{-\beta r}) + O(2^{-\beta(r+1)})). \end{aligned}$$

Hence the sum in (6) is finite and the proof is completed.

4. A proof of Theorem 1 and related results. First we introduce some notation. Following Zygmund's notation [6, pp. 42–49] we define Λ_α^p as the set of functions f in $L^p(0, 1)$, $1 \leqq p \leqq \infty$ for which

$$\sup_{0 \leqq h \leqq \delta} \|f(x+h) - f(x)\|_p = O(\delta^{-\alpha}).$$

In this notation the Hölder condition of order α on f is equivalent to f's being in Λ_α^∞. We can now prove the following theorem.

THEOREM 4. *Let $f(t)$ be a periodic function of period 1, a_n the nth Fourier coefficient of f. If $\int_0^1 f(t)dt = 0$, and if either*

(7)
a. $\sum_{k=-\infty}^{\infty} |a_{(2k+1)2^i}|^2 = O(2^{-\alpha i})$ *for some* $\alpha > 0$, *or*
b. *f is in Λ_α^2 for some $\alpha > 0$, or*
c. *f is in Λ_α^p for some $p \geqq 1$, and $\alpha > \frac{1}{2}$, or*
d. *f is in Λ_α^p for some $p \geqq 1$, and $\alpha > 0$,*
and the Fourier series of f is lacunary;

then the conclusion of Theorem 1 holds, that is, (1) is a necessary and *sufficient condition for* (2). (*Note that Theorem* 1 *is just case* c *above with* $p=\infty$.)

PROOF. This theorem follows from Theorem 3 applied to the space $L^2(0, 1)$ and the operator S which maps $f(t)$ to $f(2t)$. The restriction $\int_0^1 f(t)dt=0$ merely restricts attention to the subspace L of $L^2(0, 1)$ described earlier on which S is a shift. Note that (1) is just the explicit form of (4) for the specific shift operator under consideration and that (7) is just the explicit form of (3). That is, f_i is the function with mth Fourier coefficient equal to a_m if $m=k2^i$ with k odd and equal to zero otherwise. Thus part a of Theorem 4 is just a direct restatement of Theorem 3. (Note that if f satisfies (7) then f is in $L^2(0, 1)$.) Hence it suffices to show that conditions b, c, and d each imply condition a. It can be shown that if f is in Λ_α^p then $a_n=O(|n|^{-\alpha})$ (e.g. Zygmund [**6**, pp. 42–49]). This fact and a direct estimate shows that d implies a. That c implies b is known (Taibleson [**5**, p. 478]). If f satisfies b then $a_n=O(|n|^{-\alpha})$ and $\sum_{n=-\infty}^{\infty}|a_n|^\beta<\infty$ for some $\beta<2$ (Zygmund [**6**, p. 296]). These two facts imply $\sum_{n=-\infty}^{\infty}|a_n|^2|n|^{2\gamma}<\infty$ for some $\gamma>0$ which implies a. Thus b implies a and the proof is finished.

Theorem 4 generalizes Theorem 1 by relaxing the Hölder condition. A second direction in which Theorem 1 can be extended is to shift operators other than $(Sf)(t)=f(2t)$ and to spaces other than L. For the operator S_n defined on $L^2(0, 1)$ by $(S_nf)(t)=f(nt)$ the generalization of Theorem 1 is immediate. If the space is the Hardy space H^2 and S is multiplication by z, i.e., $(Sf)(z)=zf(z)$, Theorem 3 gives a (disappointingly trivial) condition for $f(z)/(1-z)$ to be in H^2 if $f(z)$ is; viz., if the power series for $f(z)$ has a radius of convergence greater than 1, then $f(z)/(1-z)$ is in H^2 if and only if $f(1)=0$. However, this trivial corollary suggests an alternate proof of Theorem 3. Identify H with $H^2(K)$, the space of K-valued analytic functions with square summable Taylor coefficients. (3) then implies that $f(z)$ is analytic in a disk of radius $r>1$. In this case $f(z)=g(z)-zg(z)$ can be solved for $g(z)$ in $H^2(K)$ if and only if $f(1)=0$. But $f(1)=0$ is equivalent to (4).

If the space is $L^2(T^n)$, L^2 of the n dimensional torus R^n/Z^n, then a fairly large class of isometries is generated by the nonsingular n by n matrices with integer coefficients acting in the following manner. Let M be such a matrix and $f(t_1, \cdots, t_n)=f(t)$ an element of $L^2(T^n)$. The mapping S_M defined by $(S_Mf)(t)=f(M(t))$ is a linear isometry on $L^2(T^n)$. It can be shown (Halmos [**2**]) that any isometry of a Hilbert space can be decomposed into the direct sum of a unitary operator and a shift operator. Theorem 3 can then be applied in a straightforward manner to the shift component. In fact, Theorem 1 is just a special case of this with $n=1$, $M=(2)$ and the subspace of $L^2(0, 1)$ on which M is a shift operator is the space we have been calling L.

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Princeton University