THE EQUATION \((I-S)g=f\) FOR SHIFT OPERATORS IN HILBERT SPACE

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1. Introduction. The purpose of this paper is to establish some conditions on \(f\) for the equation \(f=g-Sg\) to be solvable for \(g\) in a Hilbert space, with \(S\) a shift operator on that space.

The following theorem was stated by R. Fortet [4] and proved by M. Kac [3].

**Theorem 1.** If \(f\) is a function periodic of period 1, satisfying in \((0, 1)\) the Hölder condition of order \(\alpha\) for some \(\alpha > \frac{1}{2}\), and further \(\int_0^1 f(t)dt = 0\), then the condition

\[
\lim_{n \to \infty} \frac{1}{n} \int_0^1 \left| \sum_{i=0}^{n} f(2^i t) \right|^2 \, dt = 0
\]

is necessary and sufficient for the existence of a function \(g\) in \(L^2(0, 1)\) such that

\[
f(t) = g(t) - g(2t) \quad \text{a.e.}
\]

Kac’s method is to find formally the Fourier coefficients of \(g\) in (2), use the Hölder condition on \(f\) and the condition (1) to estimate the size of these coefficients and show that they are, in fact, the Fourier coefficients of a function in \(L^2(0, 1)\).

Z. Ciesielski has since extended the theorem to the case \(\alpha > 0\) [1]. The basic difference between Ciesielski’s approach and Kac’s is the use of Fourier-Haar coefficients in place of Fourier coefficients.

It will be the purpose of this paper to prove a theorem about inverting certain operators on Hilbert space. Theorem 1 will then be shown to be an immediate consequence of our more general theorem. First however, we introduce some definitions and established results.

2. Shift operators. Let \(H\) be a Hilbert space. We define a shift operator on \(H\) as a linear isometry on \(H\) which is unitary on no non-trivial subspace.

As an example of a shift operator let \(L\) be the subspace of \(L^2(0, 1)\) of functions whose zeroth Fourier coefficient vanishes, \(S\) the linear operator on \(L\) defined by \((Sf)(t) = f(2t)\). In this context, Theorem 1 becomes a set of conditions on \(f\) for the existence of an element

Received by the editors November 19, 1966.

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g = (I - S)^{-1}f in the Hilbert space L with S a specific shift operator on L.

We now introduce some notation which will remain fixed for the rest of the paper. S will be a shift operator on the Hilbert space H. K will be the orthogonal complement of the range of S. For any non-negative integer j let P_j be the projection from H into the subspace S^j(K), P_j = \sum_{i=0}^{j} P_i, and R_j = I - P_j (I, the identity on H). To shorten formulas, for f in H, and only for f, we shall write P_j(f) = f_j, P^i(f) = f^i, and R_j(f) = R^i. The following standard result will be used (Halmos [2]).

Theorem 2. If S is a shift operator on the Hilbert space H, and K is the orthogonal complement of the range of S, then H = K \oplus S(K) \oplus S^2(K) \oplus \cdots , and S^i(K) \perp S^j(K) for i \neq j.

A large number of simple facts such as f = \lim f^i and (f^i, S^{i+1}f) = 0 follow immediately from the decomposition of H and our notation. Such results will be used throughout the paper, often without mention.

3. The main theorem. We can now state the main result of this paper.

Theorem 3. With the above notation, let S be a shift operator on the Hilbert space H and f an element of H. If for some \beta > 0

\begin{equation}
\|f_i\| = \theta(2^{-\beta i})
\end{equation}

then a necessary and sufficient condition for the existence of a g in H with (I - S)g = f is

\begin{equation}
\lim_{n \to \infty} \frac{1}{n} \left\| \sum_{i=0}^{n} S^i f \right\|^2 = 0.
\end{equation}

The formal expansion \((I - S)^{-1} = \sum_{i=0}^{\infty} S^i (\sum_{j=0}^{\infty} f_j)\) suggests that g = \sum_{i=0}^{\infty} S^i (\sum_{j=0}^{\infty} f_j). Unfortunately this series does not converge. The proof of Theorem 3 involves rearranging this series to g = \sum_{i=0}^{\infty} (\sum_{j=0}^{\infty} S^k f_{r-k}) and showing that the rearranged series converges to the desired element. An alternate proof of Theorem 3 is outlined in the next section. Before proving the theorem we will prove four lemmas. The first two were proved by Kac [3] for the specific case of Theorem 1.

Lemma 3.1. If f satisfies (3) then (f, S^k f) = O(2^{-\beta k}).

Proof. We know that f = \sum_{i=0}^{\infty} f_i and that (f_i, S^k f_j) = 0 if k + j \neq i, hence
\[ |(f, S^k f)| = \left| \sum_{i=k}^{\infty} (f_i, S^k f_{i-k}) \right| \leq \sum_{i=k}^{\infty} \|f_i\| \|S^k f_{i-k}\| \]
\[ = \sum_{i=k}^{\infty} \|f_i\| \|f_{i-k}\| \leq \left( \sum_{i=k}^{\infty} \|f_{i-k}\|^2 \right)^{1/2} \left( \sum_{i=k}^{\infty} \|f_i\|^2 \right)^{1/2} \]
\[ \leq \|f\| \left( \sum_{i=k}^{\infty} O(2^{-\beta i}) \right)^{1/2} = O(2^{-\beta k}) \]

as was to be shown.

**Lemma 3.2.** If \( f \) satisfies (3) then
\[
\lim_{n \to \infty} \frac{1}{n} \left\| \sum_{k=0}^{n} S^k f \right\|^2 = \|f\|^2 + 2 \text{ Re} \sum_{k=1}^{\infty} (f, S^k f).
\]

**Proof.**
\[
\frac{1}{n} \left\| \sum_{k=0}^{n} S^k f \right\|^2 = \frac{1}{n} \left( \sum_{i=0}^{n} (S^i f, S^i f) + 2 \text{ Re} \sum_{0 \leq i < j \leq n} (S^i f, S^j f) \right)
\]
\[
= \|f\|^2 + \frac{2}{n} \text{ Re} \sum_{0 \leq i < j \leq n} (f, S^{i+j} f)
\]
\[
= \|f\|^2 + \frac{2}{n} \text{ Re} \left( \sum_{r=1}^{n} (n - r + 1)(f, S^r f) \right)
\]
\[
= \|f\|^2 + 2 \text{ Re} \sum_{r=1}^{n} (f, S^r f) - \frac{2}{n} \text{ Re} \sum_{r=1}^{n} (r - 1)(f, S^r f)
\]

but by the estimate of Lemma 3.1, both sums in the previous line converge as \( n \) goes to infinity. Hence \( 2/n \) times the last sum goes to zero. Therefore, taking the limit as \( n \) goes to infinity at the beginning and end of the above equation we have the desired result.

**Lemma 3.3:** If \( f \) satisfies (3) then \( \|R_j\| = \theta(2^{-\beta j}) \).

**Proof:** By the definition of \( R_j \), (3), and the subspace decomposition of \( H \) we have
\[
\|R_j\| = \left( \sum_{i=j+1}^{\infty} \|f_i\|^2 \right)^{1/2} = \left( \sum_{i=j+1}^{\infty} O(2^{-\beta i}) \right)^{1/2} = \theta(2^{-\beta j}).
\]

**Lemma 3.4:** For any \( f \) in \( H \)
\[
\left\| \sum_{i=0}^{r} S^i f_{r-i} \right\|^2 = \lim_{n \to \infty} \frac{1}{n} \left\| \sum_{i=0}^{n} S^i f \right\|^2.
\]

**Proof.** The identity
\[ \| f^r \|^2 + 2 \operatorname{Re} \sum_{k=1}^{\infty} (f^r, S^k f^r) = \left\| \sum_{i=0}^{r} S^i f_{r-i} \right\|^2 \]

can be verified by direct computation. This identity and Lemma 3.2 (applied to \( f^r \)) together imply the desired result.

Proof of Theorem 3. Throughout the proof, condition (3) will be assumed without being explicitly mentioned. The implication one way, when \( g \) is known to exist, is trivial. For the implication in the other direction define \( g_r = \sum_{k=0}^{\infty} S^k f_{r-k} \). Clearly \( g_r \) is in \( S^r(K) \). We shall show that \( \sum_{k=0}^{\infty} |g_k|^2 < \infty \). Hence \( g = \sum_{k=0}^{\infty} g_k \) is in \( H \). From the definition, it is clear that \( g \) has the desired property.

By the parallelogram law we have the following identity:
\[
\frac{2}{n} \left\| \sum_{i=0}^{n} S^i f^r \right\|^2 = \frac{1}{n} \left\| \sum_{i=0}^{n} S^i f \right\|^2 + \frac{1}{n} \left\| \sum_{i=0}^{n} S^i f^r - \sum_{i=0}^{n} S^i R_r \right\|^2
\]
\[
- \frac{2}{n} \left\| \sum_{i=0}^{n} S^i R_r \right\|^2.
\]

Letting \( n \) go to infinity in the above and using Lemma 3.4 and condition (4) of the theorem this becomes
\[
2\| g_r \|^2 = \lim_{n \to \infty} \frac{1}{n} \left\| \sum_{i=0}^{n} S^i (f^r - R_r) \right\|^2 - 2 \lim_{n \to \infty} \frac{1}{n} \left\| \sum_{i=0}^{n} S^i R_r \right\|^2.
\]

Lemma 3.2 applies to both \( (f^r - R_r) \) and \( R_r \), hence
\[
2\| g_r \|^2 = \| f^r - R_r \|^2 - 2\| R_r \|^2
\]
\[
+ 2 \operatorname{Re} \sum_{k=1}^{\infty} ((f^r - R_r, S^k(f^r - R_r)) - 2(R_r, S^k R_r))
\]
\[
= \| f^r \|^2 - \| R_r \|^2 + 2 \operatorname{Re} \sum_{k=1}^{\infty} \left[ (f^r, S^k f^r) - (f^r, S^k R_r) - (R_r, S^k f^r) \right.
\]
\[
\left. + (R_r, S^k R_r) - 2(R_r, S^k R_r) \right]
\]
\[
= \| f^r \|^2 - \| R_r \|^2 + 2 \operatorname{Re} \sum_{k=1}^{\infty} (f^r, S^k f^r) - 2 \operatorname{Re} \sum_{k=1}^{\infty} (R_r, S^k f^r).
\]

But by Lemmas 3.2 and 3.4
\[
\| f^r \|^2 + 2 \operatorname{Re} \sum_{k=1}^{\infty} (f^r, S^k f^r) = \lim_{n \to \infty} \frac{1}{n} \left\| \sum_{k=0}^{n} S^k f^r \right\|^2
\]
\[
= \left\| \sum_{i=0}^{r} S^i f_{r-i} \right\|^2 = \| g_r \|^2.
\]
Substituting this equality in the previous equation we have
\[ ||g_r||^2 = -||R_r||^2 - 2 \Re \sum_{k=1}^{\infty} (R_r, S^k f). \]
we want to show \( \sum_{r=0}^{\infty} ||g_r||^2 < \infty \). By direct application of Lemma 3.3, \( \sum_{r=0}^{\infty} ||R_r||^2 < \infty \). Therefore we need only show
\[
(6) \quad \sum_{r=0}^{\infty} \left| \sum_{k=1}^{\infty} (R_r, S^k f) \right| < \infty.
\]
Using Lemma 3.3 and the fact that \( (R_k - R_r, S^k f) = 0 \) if \( k > r \), we see
\[
\sum_{k=1}^{\infty} (R_r, S^k f) = \sum_{k=1}^{r} (R_r, S^k f) + \sum_{k=r+1}^{\infty} (R_k, S^k f) \leq \sum_{k=1}^{r} \| R_k \| \| S^k f \| + \sum_{k=r+1}^{\infty} \| R_k \| \| S^k f \|
\]
\[
= \| f \| \left( r \| R_r \| + \sum_{k=r+1}^{\infty} \| R_k \| \right)
\]
\[
= \| f \| (rO(2^{-\beta r}) + O(2^{-\beta(r+1)})).
\]
Hence the sum in (6) is finite and the proof is completed.

4. A proof of Theorem 1 and related results. First we introduce some notation. Following Zygmund’s notation [6, pp. 42–49] we define \( \Lambda_\alpha^p \) as the set of functions \( f \) in \( L^p(0, 1) \), \( 1 \leq p \leq \infty \) for which
\[
\sup_{0 \leq h \leq \delta} \| f(x + h) - f(x) \|_p = O(\delta^{-\alpha}).
\]
In this notation the Hölder condition of order \( \alpha \) on \( f \) is equivalent to \( f \)'s being in \( \Lambda_\alpha^\infty \). We can now prove the following theorem.

Theorem 4. Let \( f(t) \) be a periodic function of period 1, \( a_n \) the nth Fourier coefficient of \( f \). If \( \int_0^1 f(t) dt = 0 \), and if either
\[
a. \quad \sum_{k=-\infty}^{\infty} \left| a_n (a_{k+1}^{2i})^2 \right|^2 = O(2^{-\alpha i}) \text{ for some } \alpha > 0, \text{ or }
\]
\[
b. \quad f \text{ is in } \Lambda_\alpha^2 \text{ for some } \alpha > 0, \text{ or }
\]
\[
c. \quad f \text{ is in } \Lambda_\alpha^p \text{ for some } p \geq 1 , \text{ and } \alpha > \frac{1}{2}, \text{ or }
\]
\[
d. \quad f \text{ is in } \Lambda_\alpha^p \text{ for some } p \geq 1 , \text{ and } \alpha > 0,
\]
and the Fourier series of \( f \) is lacunary;

then the conclusion of Theorem 1 holds, that is, (1) is a necessary and
sufficient condition for (2). (Note that Theorem 1 is just case c above with \( p = \infty \).)

Proof. This theorem follows from Theorem 3 applied to the space \( L^2(0, 1) \) and the operator \( S \) which maps \( f(t) \) to \( f(2t) \). The restriction \( \int_0^1 f(t) dt = 0 \) merely restricts attention to the subspace \( L \) of \( L^2(0, 1) \) described earlier on which \( S \) is a shift. Note that (1) is just the explicit form of (4) for the specific shift operator under consideration and that (7) is just the explicit form of (3). That is, \( f_i \) is the function with \( m \)th Fourier coefficient equal to \( a_m \) if \( m = k2^i \) with \( k \) odd and equal to zero otherwise. Thus part a of Theorem 4 is just a direct restatement of Theorem 3. (Note that if \( f \) satisfies (7) then \( f \) is in \( L^2(0, 1) \).) Hence it suffices to show that conditions b, c, and d each imply condition a. It can be shown that if \( f \) is in \( \Lambda^2 \) then \( a_n = O(\langle n \rangle^{-\alpha}) \) (e.g. Zygmund [6, pp. 42–49]). This fact and a direct estimate shows that d implies a. That c implies b is known (Taibleson [5, p. 478]). If \( f \) satisfies b then \( a_n = O(\langle n \rangle^{-\alpha}) \) and \( \sum_{n=-\infty}^{\infty} |a_n|^\beta < \infty \) for some \( \beta < 2 \) (Zygmund [6, p. 296]). These two facts imply \( \sum_{n=-\infty}^{\infty} |a_n|^2 |n|^{2\gamma} < \infty \) for some \( \gamma > 0 \) which implies a. Thus b implies a and the proof is finished.

Theorem 4 generalizes Theorem 1 by relaxing the Hölder condition. A second direction in which Theorem 1 can be extended is to shift operators other than \( (Sf)(t) = f(2t) \) and to spaces other than \( L \). For the operator \( S_n \) defined on \( L^2(0, 1) \) by \( (S_nf)(t) = f(nt) \) the generalization of Theorem 1 is immediate. If the space is the Hardy space \( H^2 \) and \( S \) is multiplication by \( z \), i.e., \((Sf)(z) = zf(z)\), Theorem 3 gives a (disappointingly trivial) condition for \( f(z)/(1-z) \) to be in \( H^2 \) if \( f(z) \) is; viz., if the power series for \( f(z) \) has a radius of convergence greater than 1, then \( f(z)/(1-z) \) is in \( H^2 \) if and only if \( f(1) = 0 \). However, this trivial corollary suggests an alternate proof of Theorem 3. Identify \( H \) with \( H^2(K) \), the space of \( K \)-valued analytic functions with square summable Taylor coefficients. (3) then implies that \( f(z) \) is analytic in a disk of radius \( r > 1 \). In this case \( f(z) = g(z) - zg(z) \) can be solved for \( g(z) \) in \( H^2(K) \) if and only if \( f(1) = 0 \). But \( f(1) = 0 \) is equivalent to (4).

If the space is \( L^2(T^n) \), \( L^2 \) of the \( n \) dimensional torus \( R^n/Z^n \), then a fairly large class of isometries is generated by the nonsingular \( n \) by \( n \) matrices with integer coefficients acting in the following manner. Let \( M \) be such a matrix and \( f(t_1, \ldots, t_n) = f(t) \) an element of \( L^2(T^n) \). The mapping \( S_M \) defined by \( (S_Mf)(t) = f(M(t)) \) is a linear isometry on \( L^2(T^n) \). It can be shown (Halmos [2]) that any isometry of a Hilbert space can be decomposed into the direct sum of a unitary operator and a shift operator. Theorem 3 can then be applied in a straightforward manner to the shift component. In fact, Theorem 1 is just a...
special case of this with $n=1$, $M=(2)$ and the subspace of $L^2(0, 1)$ on which $M$ is a shift operator is the space we have been calling $L$.

References