AN ABSTRACT OSCILLATION THEOREM

KURT KREITH

Questions of oscillatory behavior are generally associated with Sturm-Liouville equations of the form

$$-(d/dt)(p(t)dx/dt) + q(t)x = \mu x \tag{1}$$

where (1) is singular at one or both ends of an interval $I=(a, b)$, $-\infty \leqq a<b\leqq \infty$. Numerous criteria exist, depending on the behavior of the coefficients $p(t)$ and $q(t)$, which assure that solutions of (1) are or are not oscillatory near a singular end point of I. However there also exist criteria for oscillatory behavior involving only the spectrum of the Sturm-Liouville operator

$$\tau = -(d/dt)(p(t)d/dt) + q(t) \tag{2}$$

in the Hilbert space $L_2(I)$ (see for example Dunford-Schwartz [**1**, Theorem XIII.7.40]). For lower semibounded operators these criteria state that if (1) is oscillatory then the essential spectrum of τ intersects $(-\infty, \mu]$, whereas if (1) is nonoscillatory, then the essential spectrum of τ does not intersect $(-\infty, \mu)$. These spectral criteria suggest that it may be of interest to generalize the notion of "oscillatory behavior" to solutions of certain operator equations of the form

$$Ax = \mu x \tag{3}$$

where A is an appropriate operator in a Hilbert space $\mathfrak{H}$ and μ is a real constant. The purpose of this paper is to examine one such generalization.

We shall assume throughout that A is a symmetric operator which is bounded below in a Hilbert space $\mathfrak{H}$. By adding sufficiently large positive multiples of x to both sides of (3), one may assume, without loss of generality, that A satisfies

$$(Ax, x) \geqq \|x\|^2 \tag{4}$$

for all x in $\mathfrak{D}_A$ (the domain of A). It will be useful to recall that a symmetric operator A satisfying (4) has a selfadjoint extension $\overline{A}$, the Friedrichs extension, obtained as follows: complete $\mathfrak{D}_A$ under the norm $|||x|||^2=(Ax, x)$ to construct a Hilbert space $\mathfrak{M}$ with inner product $((\cdot,\cdot))$; define $\mathfrak{D}_{\overline{A}}$ to consist of those elements $y\in\mathfrak{H}$ for which there exists a sequence x_n in $\mathfrak{D}_A$ such that

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$$\lim_{n\to\infty} \|y - x_n\| = 0 \quad \text{and} \quad \lim_{m,n\to\infty} |||x_m - x_n||| = 0.$$

Then, according to Friedrichs' theorem, there exists a unique selfadjoint extension $\bar{A}$ of A for which $\mathfrak{D}_{\bar{A}} \subset \mathfrak{M}$ and $((x, y)) = (x, Ay)$ for all $x \in \mathfrak{M}$ and all $y \in \mathfrak{D}_{\bar{A}}$. If τ is nonsingular on $I = (a, b)$ and $\mathfrak{D}_\tau$ is taken to be the class of infinitely differentiable functions with compact support on I, then $\bar{\tau}$ will be the selfadjoint extension of τ corresponding to the boundary conditions $x(a) = x(b) = 0$.

Let $\mathfrak{H}_k$ be a closed linear subspace of $\mathfrak{H}$ for which $\mathfrak{D}_A \cap \mathfrak{H}_k$ is dense in $\mathfrak{H}_k$ and let P_k denote the projection operator with range $\mathfrak{H}_k$. In $\mathfrak{H}_k$ we define a symmetric operator A_k as follows:

(i) $x \in \mathfrak{D}_{A_k}$ if $x \in \mathfrak{D}_A$ and $P_k x = x$,

(ii) if $x \in \mathfrak{D}_{A_k}$, then $A_k x = P_k A x$.

Since $A_k x \in \mathfrak{H}_k$ for all $x \in \mathfrak{D}_{A_k}$, it is clear that A_k is a symmetric operator in $\mathfrak{H}_k$, satisfies (4), and therefore has a Friedrichs extension, to be denoted by $\bar{A}_k$. Norms and inner products in $\mathfrak{H}_k$ will be denoted by a subscript k outside the norm or inner product symbol.

Consider now the operator equation

$$A^* x = \mu x, \tag{5}$$

where A^* is the adjoint of A.

Definition 1. We say that $\mathfrak{H}_k$ is a nodal domain for (5) if $Ax = P_k A x$ for all $x \in \mathfrak{H}_k \cap \mathfrak{D}_A$ and

$$\inf_{x \in \mathfrak{D}_{\bar{A}_k}} \frac{(\bar{A}_k x, x)_k}{\|x\|_k^2} = \mu,$$

where this infimum is achieved by an eigenfunction $u_k \in \mathfrak{D}_{\bar{A}_k}$.

Definition 2. We say that a solution x_0 of (5) is oscillatory if there exists a decomposition of $\mathfrak{H}$ into orthogonal closed subspaces.

$$\mathfrak{H} = \sum_{k=0}^{\infty} \oplus \mathfrak{H}_k \tag{6}$$

such that for $k \geqq 1$ each $\mathfrak{H}_k$ is a nodal domain for (5) and $P_k x_0$ is the required eigenfunction satisfying $(\bar{A}_k P_k x_0, P_k x_0)_k = \mu \|P_k x_0\|_k^2$. (The fact that we do not impose any conditions on $\mathfrak{H}_0$ corresponds to a lack of boundary conditions in (1).)

In one direction these definitions lead to spectral criteria for oscillatory behavior of (5) very similar to those which exist for (1).

Theorem. *If (5) has an infinite number of orthogonal nodal domains, then the essential spectrum of $\bar{A}$ intersects $[1, \mu]$.*[1]

[1] Since $(\bar{A}x, x) \geqq \|x\|^2$, we know that the spectrum of $\bar{A}$ cannot intersect $(-\infty, 1)$.

PROOF. We shall show that given any $\epsilon>0$ and any positive integer N, there exists a set of N linearly independent vectors $v_1, \cdots, v_N$ in $\mathfrak{D}_A$ such that for any nontrivial linear combination $v=c_1v_1+\cdots+c_Nv_N$, we have $(Av, v)<(\mu+\epsilon)\|v\|^2$. According to a principle adapted by Friedrichs [2], this implies either that $\bar{A}$ has at least N eigenvalues in $[1, \mu+\epsilon)$ or else that the spectrum of $\bar{A}$ is not discrete below $\mu+\epsilon$; since ϵ is arbitrarily small, N is arbitrarily large and $\bar{A}$ is bounded below, such a construction implies that the spectrum of $\bar{A}$ is not discrete in $[1, \mu]$.

To construct the set $v_1, \cdots, v_N$, let $u_1, \cdots, u_N$ denote the normalized eigenfunctions of $\bar{A}_1, \cdots, \bar{A}_N$ corresponding to the eigenvalue μ. By the definition of $\bar{A}_k$, for every u_k there exists a sequence v_{kl} satisfying

(i) $v_{kl}\in\mathfrak{D}_A$; $k=1, \cdots, N$; $l=1, 2, \cdots$;
(ii) $P_kv_{kl}=v_{kl}$; $k=1, \cdots, N$; $l=1, 2, \cdots$;
(iii) $\lim_{l\to\infty}|||P_kv_{kl}-u_k|||_k=0$;
(iv) $\lim_{l\to\infty}\|P_kv_{kl}-u_k\|_k=0$.

For $k=1, \cdots, N$ we have

$$|||v_{kl}||| = |||P_kv_{kl}|||_k \leqq |||P_kv_{kl} - u_k|||_k + |||u_k|||_k$$

and

$$|||u_k|||_k = \mu^{1/2}\|u_k\|_k \leqq \mu^{1/2}(\|u_k - P_kv_{kl}\|_k + \|v_{kl}\|).$$

Combining these inequalities

$$|||v_{kl}||| \leqq |||P_kv_{kl} - u_k|||_k + \mu^{1/2}(\|u_k - P_kv_{kl}\|_k + \|v_{kl}\|).$$

In light of (iii) and (iv) and the fact that $\lim_{l\to\infty}\|v_{kl}\|=1$, we can choose l_0 sufficiently large so that

$$|||v_{kl_0}||| < (\mu + \epsilon)^{1/2}\|v_{kl_0}\|; \qquad k = 1, 2, \cdots, N.$$

Defining $v_k=v_{kl_0}$, and noting that

$$(v_j, v_k) = 0 \quad \text{for } j \neq k; \qquad (Av_j, v_k) = 0 \quad \text{for } j \neq k;$$
$$(Av_k, v_k) < (\mu + \epsilon)\|v_k\|^2,$$

it follows that for any $v=c_1v_1+\cdots+c_Nv_N\neq 0$

$$(Av, v) = \sum_{k=1}^{N} |c_k|^2 (Av_k, v_k) < (\mu + \epsilon) \sum_{k=1}^{N} |c_k|^2\|v_k\|^2 = (\mu + \epsilon)\|v\|^2,$$

which completes the proof.

As an immediate consequence we have the following.

COROLLARY. *If $A^*x=\mu x$ has an oscillatory solution, then the spectrum of $\bar{A}$ is not discrete in $[1, \mu]$.*

If the operator A has finite deficiency indices, then the essential spectrum of all selfad'oint extensions is the same, and this fact enables one to formulate criteria in terms of the essential spectrum of A instead of $\overline{A}$. This remark applies in particular to the operator τ whose deficiency indices are at most 2.

For the operator τ we also have the following converse of Theorem 1. If (1) has a nonoscillatory solution, then the essential spectrum of τ does not intersect $[1, \mu)$. That such a converse does not hold in our more general setting is indicated by an elementary example.

Consider $Ax \equiv -\partial^2 x/\partial s^2 - \partial^2 x/\partial t^2 + q(t)x$, where $q(t)$ is chosen such that the singular selfadjoint Sturm-Liouville system

$$-d^2y/dt^2 + qy = \lambda y; \qquad 0 < t \leqq 1,$$
$$y(1) = 0,$$

has a spectrum composed of an eigenvalue at $\lambda = -1$, some continuous spectrum in the interval $(-3, -2)$, a finite number of eigenvalues for $\lambda < -3$, and an arbitrary mixture of discrete and continuous spectrum for $\lambda > -1$. Defining $R = \{(s, t) \mid 0 < s < \pi; \quad 0 < t < 1\}$ and setting $\mathfrak{H} = L_2(R)$ we can solve the boundary value problem

$$Ax = 0 \quad \text{in } R,$$
$$x = 0 \quad \text{on } \partial R \cap \{(s, t) \mid t > 0\}$$

by separation of variables. Denoting this boundary value problem by $A^*x = 0$ and setting $x(s, t) = S(s)T(t)$, we get

$$S(s) = \sin ns, \qquad -T'' + qT = -n^2 T$$

for $n = 1, 2, \cdots$. Setting $n = 1$, it follows that $A^*x = 0$ has a square integrable oscillatory solution; however, for $n > 1$ no solution of $A^*x = 0$ is oscillatory.

This example suggests that a satisfactory converse to Corollary 1 is likely to be elusive in any abstract setting general enough to include singular elliptic operators.

Bibliography

1. N. Dunford and J. T. Schwartz, *Linear operators*, Part II, Interscience, New York, 1963.

2. K. O. Friedrichs, *Criteria for discrete spectra*, Comm. Pure Appl. Math. **3** (1950), 439–449.

University of California, Davis