NONGENERATORS OF RINGS

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The purpose of this note is to examine the role of nongenerators in the theory of rings, i.e. the elements \( x \) of a ring \( R \) such that for each subset \( M \) of \( R \) for which \( R = \langle x, M \rangle \), then \( \langle M \rangle = R \). The approach used considers a ring as a group with multiple operators and consequently an ideal \( A \) generated by a subset \( S \) implies that \( S \subseteq A \). These results will include those of L. Fuchs [1] and A. Kertesz [2] whenever the ring has unity.

Unless otherwise indicated, the terminology and the necessary known results may be found in N. McCoy’s text [3].

Denote the ideal (right ideal) generated by the set \( M \) of \( R \) by \( \langle M \rangle \) (\( (M)r \)).

**Definition.** An element \( x \in A \) is a generator of an ideal (right ideal) \( A \) in a ring \( R \) provided that there is a subset \( M \) of \( A \) such that \( A = \langle x, M \rangle \) \( (A = \langle x, M \rangle_r) \) and \( \langle M \rangle \subseteq A \) \( (\langle M \rangle_r \subseteq A) \) properly. Otherwise \( x \) is called a nongenerator of \( A \). (Note that \( M \) may be empty.)

The set of nongenerators of an ideal (right ideal) \( A \) in a ring \( R \) will be denoted by \( \Phi (\Phi_r) \), respectively.

Immediate consequences of the definition are the following:
(i) For an element \( x \) of a ring \( R \), \( x \in \Phi (x \in \Phi_r) \) if and only if \( \langle x \rangle \subseteq \Phi (\langle x \rangle_r \subseteq \Phi_r) \).
(ii) In a ring \( R \), \( \Phi \) is an ideal and \( \Phi_r \) is a right ideal.

Throughout this paper a maximal ideal of a ring \( R \) will be a proper ideal of \( R \) that is not contained in another proper ideal of \( R \). Similarly for maximal right (left) ideals.

(iii) In a ring \( R \), \( \Phi (\Phi_r) \) is the intersection of the maximal ideals (right ideals), if they exist, and is \( R \) otherwise.
(iv) For a ring \( R \) and homomorphism \( \theta \) of \( R \), \( \Phi \theta \subseteq \Phi (R \theta) \) and \( \Phi_r \theta \subseteq \Phi_r (R \theta) \).
(v) For an ideal \( A \) of a ring \( R \), \( A \subseteq \Phi \) implies that \( \Phi (R/A) = \Phi /A \) and \( A \subseteq \Phi_r \) implies that \( \Phi_r (R/A) = \Phi_r /A \).
(vi) For a ring \( R \), if \( A \) is an ideal (right ideal) of \( R \), then \( \Phi (A) \subseteq \Phi (\Phi_r (A) \subseteq \Phi_r) \).
(vii) In a ring \( R \), \( \Phi = \langle 0 \rangle \) (\( \Phi_r = \langle 0 \rangle \)) implies that \( \Phi (A) = \langle 0 \rangle \) (\( \Phi_r (A) = \langle 0 \rangle \)) for each ideal (right ideal) \( A \) of \( R \). Such rings will be called \( \Phi \)-free or \( \Phi_r \)-free respectively.

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(viii) In a ring $R$, $A = \Phi(A)$ ($A = \Phi_r(A)$) for an ideal (right ideal) $A$ of $R$ implies that $A \subseteq \Phi_r(A)$.

(ix) If $R$ is a ring and $R = M_1 \oplus \cdots \oplus M_n$, then $\Phi = \Phi(M_1) \oplus \cdots \oplus \Phi(M_n)$ for ideals $M_i$ of $R$.

(x) In a ring $R$, if $A$ is a minimal ideal (right ideal) such that $A \uplus \Phi_r(A)$, then there exists a maximal ideal (right ideal) $B$ such that $R = A \oplus B$.

(xi) If $R$ is a zero ring ($R^2 = (0)$), then $\Phi = \Phi(R^+)$, $\Phi(R^+)$ the Frattini subgroup of the additive group $R^+$.

In the remaining portion of this note, the Jacobson radical and the upper Baer radical will be denoted by $J$ and $N$ respectively.

**Theorem 1.** In a ring $R$, $\Phi_r \subseteq J$ and $\Phi \subseteq N$.

**Proof.** If $J \neq R$, then $J$ is the intersection of the modular maximal right ideals of $R$; and if $N \neq R$, then $N$ is the intersection of the modular maximal ideals.

Note that in Theorem 1 equality may not occur as the ring $\{0, 2; \text{mod } 4\}$ exemplifies.

**Theorem 2.** In a ring $R$, $RJ \subseteq \Phi_r$ and $JR \subseteq \Phi_l, \Phi_l$ denoting the set of nongenerators with respect to left ideals.

**Proof.** Since the result follows if $\Phi_r = R$, consider the case that $\Phi_r \subseteq R$ properly. Suppose there is an element $x \in R$ such that $yx \notin \Phi_r$ for some element $y \in R$. Then there exists a maximal right ideal $M$ such that $yx \notin M$. $M$ defines a simple right $R$-module $R/M \cong R^*$, and under the natural $R$-homomorphism $\theta$ of $R \to R^*$, $y \theta \neq 0$ and $(yx) \theta \neq 0$. So $(R^*)R = R^*$, and an element $z \in R$ exists such that $(yxz) \theta = y \theta$. Then note that if $xz$ is r.q.r., there exists an element $b \in R$ such that $xz + b = xzb$. Under $\theta$, $yxz + by = yxz$ becomes $(yxz) \theta = -(yb) \theta + (yxb) \theta = -(yb) \theta + (yb) \theta = 0$. So $yxz \notin M$ and a contradiction. Therefore $xz$ cannot be r.q.r. In conclusion, if $x$ has the property that $yx \notin \Phi_r$ for some $y \in R$, then $x \in J$. So for each element $x \in J$, $Rx \subseteq \Phi_r$, i.e. $RJ \subseteq \Phi_r$. Similarly $JR \subseteq \Phi_l$. (Note: this proof was suggested by a result of Kertesz [2].)

**Corollary 2.1.** (a) For a ring $R$, $\Phi_r$ and $\Phi_l$ are ideals in $R$.

(b) For a ring $R$, $J^2 \subseteq \Phi_r \cap \Phi_l$.

(c) $J = (\Phi_r: R) = (\Phi_l: R)$

(d) For a ring $R$, $x \in J$ iff $R \times R \subseteq \Phi_r \cap \Phi_l$.

Since in general both $\Phi_r$ and $\Phi_l$ are in $J$, it follows that in each primitive ring the right ideals and the left ideals are $\Phi_r$- and $\Phi_l$-free respectively. If the ring is a simple nonradical ring, then the ring is
Φ-free. For the simple primitive rings, all three hold. And for a field $F$, $Φ(F) = (0)$.

In general $Φ ⊆ J$. For example: let $R$ be the ring of all linear transformations of a vector space $V$ with a denumerable basis. It is known (e.g., see [3]) that $R$ is a primitive ring and $J = (0)$. Since $R$ has unity, $N = R$; and, in fact, the only proper ideal besides $(0)$ is the ideal of elements of finite rank. This ideal is $N = Φ$. Also note that $Φ_r = Φ_t = (0)$.

**Theorem 3.** For a ring $R$ having $R^2 = R$, $Φ$ is a semiprime ideal.

**Proof.** Each maximal ideal is prime. If $A$ is an ideal for which $A^2 ⊆ Φ$, then $A^2$ is contained in each maximal ideal $M$. So $A$ is contained in each $M$. Therefore $A ⊆ Φ$.

**Corollary 3.1.** For a ring $R$ having $R^2 = R$, the prime radical is contained in $Φ$.

**Corollary 3.2.** For a ring $R$ having $R^2 = R$, $J ⊆ Φ$ iff $J^2 ⊆ Φ$.

**Theorem 4.** For a ring $R$ having $R^2 = R$ and center $Z$, $N ∩ Z ⊆ Φ$, and $N ∩ Z = Φ$.

**Proof.** If $A = N ∩ Z ⊆ Φ$, and $M$ is a maximal right ideal not containing $A$, then $R = A + M$. This implies that $M$ is a maximal ideal, and $R^2 = R$ implies that $R/M$ is a simple commutative nonzero ring. Hence $M$ is modular and $N ⊆ M$ implies that $A ⊆ M$. So $A ⊆ Φ$. Similarly $N ∩ Z ⊆ Φ$.

**Corollary 4.1.** If $R$ is a commutative ring and $R^2 = R$, then $J = Φ$.

**Theorem 5.** For a ring $R$ having $R^2 = R$, $Φ_r = Φ_t = J$.

**Proof.** Consider $Φ_r$, and note that for $R = Φ_r$ and $Φ_r ⊆ J$ implies that $J = Φ_r$. So then consider the case that $Φ_r ⊆ R$ properly. By Theorem 2, $J^2 ⊆ Φ_r$. Form $R/J^2 ∼= R^*$ noting that $J^* ∼= J(R/J^2) = J/J^2$ and that $Φ^* ∼= Φ_r(R/J^2) = Φ_r/J^2$. If $x ∊ J^*$ and $x ∊ Φ^*$, there exists a maximal right ideal $M^*$ such that $x ⊆ M^*$. Under the natural $R^*$-homomorphism $θ$ of $R^* → R^*/M^*$, $R^*$ is mapped onto a simple right $R^*$-module $R^*/M^*$. Since $x ∊ M^*$, then $J^*θ = R^*/M^*$. But $J^*θ = (0)$ implies that $R^*/M^*$ is annihilated by $R^*$, i.e. $(R^*/M^*)R^* = (0)$. This contradicts the hypothesis that $R^2 = R$ since, in turn, this implies that $R^{*2} = R^*$ and $(R^*/M^*)R^* = R^*/M^*$. So $J^* ⊆ M^*$. This leads to $J^* ⊆ Φ^*$ and hence $J ⊆ Φ_r$. So the result follows.

Similarly $Φ_t = J$.

**Corollary 5.1 (L. Fuchs [1]).** For a ring with unity, $Φ_r = Φ_t = J$. 


Theorem 6. If $R$ satisfies the d.c.c. on right ideals, then $\Phi = (0)$ if and only if $R$ is a direct sum of a finite collection of simple ideals.

Proof. Consider the intersections of all finite collections of maximal ideals. By the d.c.c. on right ideals, each linear system has a minimal element, say $D$. If $M$ is a maximal ideal, then $D = D \cap M$. So $D \subseteq \Phi$ and $D = (0)$. As is known, if there exists in a ring a finite number of maximal ideals $M_i, (i = 1, \ldots, n)$ with zero intersection, then $R$ is isomorphic to the direct sum of some or all of the simple rings $R/M_i (i = 1, \ldots, n)$. By (ix) each direct summand has $\Phi(R/M_i) = (0)$ since $R/M_i$ is a simple ideal.

Again by (ix) the converse is evident.

Theorem 7. If $R$ is a ring with d.c.c. on right ideals, then both $\Phi$ and $\Phi_i$ are contained in $\Phi$.

Proof. The theorem is valid whenever $R = \Phi$, so consider the case that $\Phi \subset R$ properly. In particular restrict attention to $R^* = R/M$ for a maximal ideal $M$. For either $R^* = (0)$ or $R^* = R^*$, $\Phi^* = (0)$. Hence under the natural homomorphism $\theta$ of $R \to R^*$, $\Phi_\theta \subseteq (0)$ implies that $\Phi_r \subseteq M$. So $\Phi_r \subseteq \Phi$ and similarly $\Phi_i \subseteq \Phi$.

Theorem 8. If $R$ is a ring with the d.c.c. on right ideals, then $\Phi_r = \Phi_i = \Phi$.

Proof. Since $\Phi_r$ is an ideal of $R$ form $R^* \cong R/\Phi_r$ having $\Phi_r^* = \Phi_r(R^*) = (0), \Phi^* \cong \Phi_r$ and $J^* \cong J/\Phi_r$. If $M^*$ is a maximal right ideal such that $\Phi^* \subseteq M^*$, then $R^* = \Phi^* + M^*$. However, since $R^* \subseteq \Phi_r^* = (0)$, then $\Phi^*$ is in the annihilator of $M^*$. This implies that $M^*$ is an ideal of $R^*$ and hence a contradiction to the assumption that $\Phi^* \subseteq M^*$. So $\Phi^* = (0)$, i.e. $\Phi \subseteq \Phi_r$, and the result follows. Similarly, $\Phi_i = \Phi$.

Theorem 9. For a ring $R$ with d.c.c. on right ideals and $R$ not a radical ring, then $\Phi = J$ if and only if $R^2 = R$.

Proof. Suppose $R^2 = R$ and there exists a maximal ideal $M$ such that $J \subseteq M$. Then under the natural homomorphism $\theta$ of $R \to R/M = R^*$, $J\theta = R^*$. However, since $J^2 \subseteq \Phi \subseteq M$ it follows that $R^* = (0)$ and this contradicts $R^2 = R$. So $J \subseteq \Phi$. Since $J = N$ and $\Phi \subseteq N$, then $\Phi = J$.

On the other hand, suppose that $J = \Phi \subset R$ properly. Form $R/\Phi \cong R^*$ and note that $J^* \cong J(R/\Phi) = (0) = \Phi^* \cong \Phi(R/\Phi)$. As is known, $J^* = (0)$ implies that $R^* = R^*$. If $R^2 \subset R$ properly and $R^* = R^*$ under the natural homomorphism $\theta$ of $R \to R^*$, then $R = \Phi + R^2 = R^2$ and a contradiction. So $R^2 = R$.
In a radical ring $R$ the condition $\Phi = J$ does not necessarily imply that $R^a = R$. For example, let $R$ be a zero ring having $R^+$ a group of type $p^\infty$. Then $\Phi(R) = \Phi(R^+) = R^+$ and $J = R$.

**Bibliography**