ON THE RESTRICT-INDUCE MAP OF GROUP CHARACTERS

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The purpose of this note is to give a new proof of the following theorem due to E. Artin.

**Theorem.** Each ordinary irreducible character of a finite group is a linear combination with rational coefficients of characters induced from linear characters of cyclic subgroups.

Let $G$ be a finite group and let $K_1, \cdots, K_n$ be the classes of conjugate elements of $G$. Then the ordinary irreducible characters $\chi_1, \cdots, \chi_n$ form a basis of the vector space $U$ of all complex-valued class functions on $G$ over the complex number field. This basis is orthonormal relative to the usual inner product. There is another orthonormal basis $\alpha_1, \cdots, \alpha_n$ defined by

$$\alpha_i(g) = 0 \quad \text{if} \quad g \in G - K_i,$$

$$= \sqrt{|C(g)|} \quad \text{if} \quad g \in K_i,$$

where $|C(g)|$ is the order of the centralizer $C(g)$ of $g$ in $G$.

Let $H$ be a subgroup of $G$. We define a linear transformation $T$ of $U$ by $T(\theta) = (\theta | H)^*$, the class function on $G$ obtained by inducing the restriction of $\theta$ to $H$. It will be noted that $T$ is symmetric, for, applying Frobenius reciprocity twice, we have

$$(T(\theta), \eta)_\sigma = ((\theta | H)^*, \eta) = (\theta | H, \eta | H)_\sigma$$

$$= (\theta, (\eta | H)^*) = (\theta, T(\eta))_\sigma.$$ 

Let

$$T(\chi_i) = \sum_j a_{ij} \chi_j$$

so that $A = (a_{ij})$ is the matrix of $T$ relative to the basis $\chi_1, \cdots, \chi_n$. It is known that each $a_{ij}$ is a nonnegative rational integer.

If $K_i \cap H$ is empty, the formula for the value of an induced character implies $T(\alpha_i) = 0$. If $K_i \cap H$ is nonempty, then

$$K_i \cap H = C_1 + \cdots + C_s$$

where each $C_j$ is a class of $H$. Let $g \in K_i$ and let $g_j \in C_j, j = 1, \cdots, s$. Then

Received by the editors December 16, 1966.
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\[ T(\alpha_i)(g) = \left[ \begin{array}{c|c|c} C(g) & |K_i \cap H| & / |H| \\ \hline \end{array} \right] \alpha_i(g) \]

\[ = \left[ \sum \begin{array}{c} C(g) \mid |H \cap C(g_j)| \end{array} \right] \alpha_i(g) \]

\[ = \left[ \sum \begin{array}{c} C(g_i : C(g_i) \cap H) \end{array} \right] \alpha_i(g). \]

Since \( T(\alpha_i)(g) = \alpha_i(g) = 0 \) for \( g \in G - K_i \), we have

\[ T(\alpha_i) = r_i \alpha_i \]

where \( r_i = \sum_j C(g_j : C(g_j) \cap H) \) is a sum of positive integers. Hence \( \alpha_1, \ldots, \alpha_n \) is a basis of eigenvectors of \( T \) and the eigenvalues \( r_1, \ldots, r_n \) are nonnegative integers. The above shows that the rank of \( T \) is the number of classes of \( G \) meeting \( H \). The kernel of \( T \) is the set of class functions on \( G \) vanishing on those classes meeting \( H \).

The two matrices of \( T, A \) and \( \text{diag} \{ r_1, \ldots, r_n \} \), are similar over the complex number field. Since their entries are rational numbers, they are already similar over the rational number field \( Q \). This implies the existence of a basis \( \theta_1, \ldots, \theta_n \) of eigenvectors of \( T \) which are \( Q \)-linear combinations of \( \chi_1, \ldots, \chi_n \) with eigenvalues \( r_1, \ldots, r_n \).

We consider \( \theta_1, \ldots, \theta_n \) as a basis of the vector space \( V(G) \) of all \( Q \)-linear combinations of \( \chi_1, \ldots, \chi_n \) and restrict \( T \) to \( V(G) \). Assume the notation is chosen so that \( r_i \neq 0 \), \( i = 1, \ldots, t \) while \( r_i = 0 \) for \( i = t+1, \ldots, n \). Note that for \( i \leq t \), \( \theta_i(g) = 0 \) if \( g \) lies in a class of \( G \) not meeting \( H \), because \( \theta_i(g) = (1/r_i) T(\theta_i)(g) = 0 \). On the other hand for \( i > t \), \( \theta_i(g) = 0 \) if \( g \) lies in a class meeting \( H \). Now let \( \chi \) be an irreducible character of \( G \) and \( a_1, \ldots, a_n \) rational numbers such that \( \chi = \sum_{j=1}^n a_j \theta_j \). Set \( \mu = \sum_{j=1}^t a_j \theta_j \). It follows that \( \mu \) vanishes on those classes not meeting \( H \) and that \( \mu \mid H = \chi \mid H \). Furthermore,

\[ \mu = \sum_{j=1}^t \frac{a_j}{r_j} \theta_j = \sum_{j=1}^t \frac{a_j}{r_j} \theta_j \mid H. \]

Since \( \theta_j \mid H \) is a \( Q \)-linear combination of irreducible characters of \( H \), we have the following proposition.

**Proposition.** Given an irreducible character \( \chi \) of \( G \) and a subgroup \( H \) of \( G \), there exists a \( Q \)-linear combination \( \mu \) of characters of \( G \) induced from irreducible characters of \( H \) such that \( \mu = 0 \) on classes of \( G \) not meeting \( H \) and \( \mu \mid H = \chi \mid H \).
The function $\mu$ is not uniquely determined. In the following paragraph we write $\chi(H)$ for one such function, indicating its dependence on the given character $\chi$ and the given subgroup $H$. The particular choice of $\chi(H)$ is immaterial.

Now let $g_i \in K_i$, $i = 1, 2, \ldots, n$ and let $H_i$ be the cyclic subgroup generated by $g_i$. Let $\chi$ be an irreducible character of $G$. The proof of the main result is completed by noting

$$\chi = \chi(H_1) + \chi(H_2) + \cdots + \chi(H_n)$$
$$- \chi(H_1 \cap H_2) - \chi(H_1 \cap H_3) - \cdots - \chi(H_{n-1} \cap H_n)$$
$$+ \chi(H_1 \cap H_2 \cap H_3) + \cdots + \chi(H_{n-2} \cap H_{n-1} \cap H_n)$$
$$\vdots$$
$$+ (-1)^{n+1} \chi(H_1 \cap H_2 \cap \cdots \cap H_n)$$

where in the $k$th row there is one and only one term for each possible intersection of $k$ of the $H_i$'s. (This expression holds whenever $H_1, \ldots, H_n$ is a set of subgroups of $G$ such that each class of $G$ meets at least one $H_i$.)

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