NTH ROOTS OF OPERATORS

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I. Introduction. Let $A$ and $E$ be continuous linear transformations which map a Banach space into itself. If $E$ commutes with $A^n$, for some integer $n$ greater than 1, we do not necessarily know whether $E$ commutes with $A$. The main purpose of this paper is to exhibit a proof of the following theorem:

Let $A$ and $E$ be continuous linear transformations which map a complex Banach space into itself. Let $n$ be an integer greater than 1. If $\sigma(A) \cap \sigma(e^{2\pi ik/n}A) = \emptyset$ for $k = 1, \ldots, n-1$ and $E$ commutes with $A^n$, then $E$ commutes with $A$.

A significant corollary of this theorem is as follows:

If $A$ is a continuous linear transformation from a Hilbert space into itself and $A^n$ is normal, then $A$ is normal, whenever the spectrum of $A$ satisfies the condition stated in the theorem.

Throughout the paper the following notation and terminology will be used. $\mathfrak{X}$ is a complex Banach space and $\mathcal{L}(\mathfrak{X})$ is the space of continuous linear transformations from $\mathfrak{X}$ into $\mathfrak{X}$. The identity element of $\mathcal{L}(\mathfrak{X})$ will be denoted by $I$. If $A \in \mathcal{L}(\mathfrak{X})$, $\sigma(A)$ will denote the spectrum of $A$ and $\rho(A)$ the resolvent set of $A$. If $C$ is a simple closed rectifiable curve in the complex plane, $C$ will be assumed to be oriented in a counterclockwise direction. By the interior of $C$ we mean the bounded component of the complement of $C$; the exterior of $C$ is the unbounded component.

II. Essential background material. The proof of the above-stated theorem relies heavily on the following results, which are found in The spectrum of linear transformations by E. R. Lorch [1]:

Theorem 1. Let $C$ be a simple closed rectifiable curve lying entirely within $\rho(A)$, where $A \in \mathcal{L}(\mathfrak{X})$. Then the contour integral

$$ P = \frac{1}{2\pi i} \int_C (zI - A)^{-1}dz $$

exists and represents an element of $\mathcal{L}(\mathfrak{X})$.

The transformation $P$ is unchanged if the curve $C$ is continuously...
deformed into a curve \( C' \), providing only that the deformation is effected without going outside of \( \rho(A) \) [1, p. 241].

**Theorem 2.** The operator \( P \) defined by the curve \( C \) as in Theorem 1 has the following properties:

(a) \( P = 0 \) if and only if every point in the interior of \( C \) lies in \( \rho(A) \).

(b) \( P = I \) if and only if every point in the exterior of \( C \) lies in \( \rho(A) \) [1, p. 244].

III. Basic lemmas. It is well known that if \( A \subseteq \mathbb{C} \), \( \sigma(A) \) is a nonempty, compact subset of the complex plane. If for some integer \( n, \ n > 1, \ \sigma(A) \cap \sigma(e^{2\pi ik/n}A) = \emptyset \) for \( k = 1, \ldots, n-1 \), then \( M > 0 \), where

\[
M = \min_{k=1, \cdots, n-1} \text{g.l.b.}\{ |z - w| : z \in \sigma(A), w \in \sigma(e^{2\pi ik/n}A) \}.
\]

Since \( \sigma(A) \) is compact, there exist \( z_1, \ldots, z_m \) in \( \sigma(A) \) such that \( \sigma(A) \subseteq \bigcup_{j=1}^{n} B(z_j; M/3) \), where \( B(z; M/3) \) is the open ball in the complex plane of radius \( M/3 \) and center \( z \). Let \( V_1, \ldots, V_p \) be the components of \( \bigcup_{j=1}^{n} B(z_j; M/3) \) and let

\[
s_p(k-1)+j = \sigma(e^{2\pi ik/n}A) \cap (e^{2\pi ik/n}V_j)
\]

for \( k = 1, \ldots, n \) and \( j = 1, \ldots, p \). Note the following facts:

1. for each \( j \), \( s_j \) is a compact subset of the complex plane;
2. for \( q \neq j \), \( s_q \cap s_j = \emptyset \);
3. for \( k = 1, \ldots, n \), \( \sigma(e^{2\pi ik/n}A) = \bigcup_{j=1}^{p} s_p(k-1)+j \).

Let \( S = \{s_k: k = 1, \ldots, np\} \). We will say that \( s_j \) is interior to \( s_q \) if \( j \neq q \) and whenever \( C \) is a simple closed rectifiable curve such that \( s_q \) is in the interior of \( C \), then \( s_j \) is also in the interior of \( C \). We will denote this by \( s_j < s_q \). Because of the way in which \( S \) was constructed, if \( s_j < s_q \), there exists a simple closed rectifiable curve \( C' \) such that \( s_j \) is in the interior of \( C' \) and \( s_q \) is in the exterior of \( C' \).

\( C \) will be called a **proper curve for \( s_j \)** if \( C \) is a simple closed rectifiable curve, lying entirely in \( \bigcap_{k=1}^{n} \rho(e^{2\pi ik/n}A) \), such that (1) \( s_j \) is in the interior of \( C \) and (2) whenever \( s_q \) is in the interior of \( C \), then \( s_q = s_j \) or \( s_q < s_j \).

An element \( s_k \) of \( S \) will be said to be of **order 0** if there exists no \( s_j \) in \( S \) such that \( s_j < s_k \). Since \( S \) is a finite set, there must exist elements of \( S \) of order 0. A **branch of length \( k \)** in \( S \) is a sequence \( \{t_j: j = 1, \ldots, k\} \) of elements of \( S \) such that (1) \( t_1 \) is of order 0, (2) \( t_j < t_{j+1} \) for \( j = 1, \ldots, k-1 \), and (3) there exists no \( s_i \) in \( S \) such that \( t_j < s_i < t_{j+1} \). An element \( s_i \) of \( S \) is of **order \( k \)**, \( k \geq 1 \), if there exists a branch \( \{t_j: j = 1, \ldots, k\} \) of length \( k \) in \( S \) such that \( t_k < s_i \) and there
exists no branch \( \{ i_j : j = 1, \ldots, k+1 \} \) of length \( k+1 \) in \( S \) such that \( i_{k+1} < s_i \).

**Lemma 1.** Let \( n \) be a positive integer. There exist nonzero complex numbers \( b_1, \ldots, b_n \) such that

\[
(\frac{1}{z^n I - A^n})^{-1} = A^{-(n-1)} \sum_{j=1}^{n} b_j (z I - e^{2\pi i j/n} A)^{-1},
\]

whenever \( A \) is an invertible element of \( \mathfrak{S}(X) \) and \( z \in \cap_{j=1}^{n} \rho(e^{2\pi i j/n} A) \).

**Proof.** Let \( \{ b_1, \ldots, b_n \} \) be the unique solution of the system of equations

\[
\sum_{k=1}^{n} b_k e^{2\pi i k(j-1)/n} = 0 \quad \text{for } j = 1, \ldots, n - 1,
\]

\[
\sum_{k=1}^{n} b_k e^{2\pi i (n-1)/n} = 1.
\]

Let \( z \) be any element of \( \cap_{j=1}^{n} \rho(e^{2\pi i j/n} A) \). Then

\[
A^{-n} = \sum_{j=1}^{n} z^{n-j} A^{j-1} \left( \sum_{k=1}^{n} b_k e^{2\pi i k(j-1)/n} \right)
\]

\[
= \sum_{k=1}^{n} b_k \left( \sum_{j=1}^{n} z^{n-j}(e^{2\pi i k/n} A)^{j-1} \right)
\]

\[
= \sum_{k=1}^{n} b_k (z I - e^{2\pi i k/n} A)^{-1}(z^n I - A^n)
\]

\[
= (z^n I - A^n) \sum_{k=1}^{n} b_k (z I - e^{2\pi i k/n} A)^{-1}.
\]

Therefore, \( (z^n I - A^n)^{-1} = A^{-(n-1)} \sum_{k=1}^{n} b_k (z I - e^{2\pi i k/n} A)^{-1} \).

To see that each of the \( b_j \) is nonzero we need only consider a special case. Let \( A = I \). Let \( C \) be a simple closed rectifiable curve such that \( e^{2\pi i j/n} \) is in the interior of \( C \) and \( e^{2\pi i k/n} \), \( k \neq j \), is in the exterior of \( C \). Then, using the last equation in the preceding paragraph, we see that

\[
\frac{1}{2\pi i} \int_C (z^n - 1)^{-1} dz = \sum_{k=1}^{n} b_k \frac{1}{2\pi i} \int_C (z - e^{2\pi i k/n})^{-1} dz = b_j.
\]

However,

\[
\frac{1}{2\pi i} \int_C (z^n - 1)^{-1} dz = \prod_{k=1; k \neq j}^{n} (e^{2\pi i j/n} - e^{2\pi i k/n})^{-1} \neq 0.
\]

Therefore, \( b_j \neq 0 \), \( j = 1, \ldots, n \).
The most serious problem which arose in the proof of Theorem 3 was the interrelation between the components of \( \sigma(A) \) and the components of \( \sigma(e^{2\pi i k/n}A) \), \( k = 1, \ldots, n-1 \). If there exists a simple closed rectifiable curve \( C \) such that \( \sigma(A) \) is interior to \( C \) and \( \sigma(e^{2\pi i k/n}A) \) is exterior to \( C \) for \( k = 1, \ldots, n-1 \), then by Theorem 2 we have

\[
\int_C (zI - e^{2\pi ik/n}A)^{-1}dz = 0 \quad \text{for} \quad k = 1, \ldots, n-1
\]

and \((1/2\pi i)\int_C (zI - A)^{-1}dz = I\). Using Lemma 1, we see that

\[
\frac{1}{2\pi i} \int_C (z^nI - A^n)^{-1}dz = A^{-(n-1)}b_n.
\]

Therefore, if \( E \) commutes with \( A^n \), \( E \) also commutes with \( A^{-(n-1)} \) and with \( A \). However, in the case in which there exists no such curve \( C \), the situation is much more complicated.

**Lemma 2.** Let \( A \) and \( E \) be elements of \( \mathcal{L}(\mathfrak{X}) \) and let \( n \) be an integer greater than 1. Suppose that

\[
\sigma(A) \cap \sigma(e^{2\pi i k/n}A) = \emptyset \quad \text{for} \quad k = 1, \ldots, n-1
\]

and that \( E \) commutes with \( A^n \). If \( s_j \) is an element of \( S \) and \( C \) is a proper curve for \( s_j \), then \( E \) commutes with

\[
A^{-(n-1)} \int_C (zI - e^{2\pi ik/n}A)^{-1}dz, \quad k = 1, \ldots, n.
\]

**Proof.** The proof of this lemma will be by induction on the order of the elements of \( S \). Note that if \( C \) is any simple closed rectifiable curve, lying in \( \bigcap_{k=1}^n \sigma(e^{2\pi i k/n}A) \), then \( \int_C (z^nI - A^n)^{-1}dz \) is an element of \( \mathcal{L}(\mathfrak{X}) \) and commutes with \( E \) [1, pp. 240–241].

Let \( s_j \) be an element of \( S \) of order 0 and let \( C \) be a proper curve for \( s_j \). Assume that \( s_j \subset \sigma(e^{2\pi i k/n}A) \). By Theorem 2,

(1) \[
\int_C (zI - e^{2\pi i q/n}A)^{-1}dz = 0 \quad \text{for} \quad q \neq k \quad \text{and} \quad q = 1, \ldots, n.
\]

Using (1) and Lemma 1, we have

(2) \[
\int_C (z^nI - A^n)^{-1}dz = b_k A^{-(n-1)} \int_C (zI - e^{2\pi ik/n}A)^{-1}dz.
\]

From (1) and (2) it follows that \( E \) commutes with

\[
A^{-(n-1)} \int_C (zI - e^{2\pi i q/n}A)^{-1}dz \quad \text{for} \quad q = 1, \ldots, n
\]
whenever $C$ is a proper curve for an element of $S$ of order 0.

Assume now that whenever $s_j$ is of order $p$, $0 \leq p \leq m$, and $C$ is a proper curve for $s_j$, $E$ commutes with

$$A^{-(n-1)} \int_C (zI - e^{2\pi i q/n} A)^{-1} dz \quad for \quad q = 1, \ldots, n.$$ 

Let $s_{j_0}$ be an element of $S$ of order $m+1$ and $C$ be a proper curve for $s_{j_0}$. Let $t_1, \ldots, t_r$ be the elements of $S$ such that $t_j < s_{j_0}, j = 1, \ldots, r$ and there exists no $s_k$ in $S$ such that $t_j < s_k < s_{j_0}$. Let $C_j, j = 1, \ldots, r$, be a proper curve for $t_j$. Since each of the $t_j, j = 1, \ldots, r$, is of order $p$, $p \leq m$, then $E$ commutes with

$$A^{-(n-1)} \int_{C_j} (zI - e^{2\pi i q/n} A)^{-1} dz \quad for \quad k = 1, \ldots, n \quad and \quad j = 1, \ldots, r.$$ 

Assume that $s_{j_0} \subset \sigma(e^{2\pi i k/n} A)$. It follows from Theorems 1 and 2 that for $q \neq k$,

$$A^{-(n-1)} \int_C (zI - e^{2\pi i q/n} A)^{-1} dz = \sum_{p=1}^r A^{-(n-1)} \int_{C_p} (zI - e^{2\pi i q/n} A)^{-1} dz.$$ 

Therefore, the induction hypothesis implies that $E$ commutes with

$$A^{-(n-1)} \int_C (zI - e^{2\pi i q/n} A)^{-1} dz \quad for \quad q \neq k \quad and \quad q = 1, \ldots, n.$$ 

Moreover, since $E$ commutes with $\int_C (z^n I - A^n)^{-1} dz$ and

$$\int_C (z^n I - A^n)^{-1} dz = \sum_{q=1}^n b_q A^{-(n-1)} \int_C (zI - e^{2\pi i q/n})^{-1} dz,$$

$E$ also commutes with $A^{-(n-1)} \int_C (zI - e^{2\pi i k/n} A)^{-1} dz$. Thus the proof by induction is complete.

IV. Theorems.

**Theorem 3.** Let $A$ and $E$ be elements of $\mathcal{L}(X)$ and let $n$ be an integer greater than 1. If $\sigma(A) \cap \sigma(e^{2\pi i k/n} A) = \emptyset$ for $k = 1, \ldots, n-1$, and $A^n$ commutes with $E$, then $A$ commutes with $E$.

**Proof.** Let $t_1, \ldots, t_r$ be the elements of $S$ which are interior to no other elements of $S$. Let $C_j, j = 1, \ldots, r$, be a proper curve for $t_j$. Let $C$ be a simple closed rectifiable curve such that $t_j$ is interior to $C$ for $j = 1, \ldots, r$. By Theorem 2

$$\frac{1}{2\pi i} \int_C (zI - A)^{-1} dz = I$$
since $\sigma(A)$ lies entirely in the interior of $C$. Moreover, by Theorem 1 and Theorem 2

$$\frac{1}{2\pi i} \int_{C} (zI - A)^{-1} dz = \sum_{j=1}^{r} \frac{1}{2\pi i} \int_{c_j} (zI - A)^{-1} dz.$$ 

Combining (1) and (2), we see that

$$\sum_{j=1}^{r} \frac{A^{-(n-1)}}{2\pi i} \int_{c_j} (zI - A)^{-1} dz = A^{-(n-1)}.$$ 

Lemma 2 asserts that $E$ commutes with each of the summands in the left hand side of equation (3). Therefore $E$ commutes with $A^{-(n-1)}$ as well as $A^n$, which implies that $E$ commutes with $A$.

One of the simplest situations in which the hypotheses of Theorem 3 are satisfied is when $X$ is a Hilbert space and either $\text{Re} A \gg p > 0$ or $\text{Im} A \gg p > 0$, for some real number $p$. Then $A$ and $A^2$ commute with exactly the same operators, since $\sigma(A) \cap \sigma(-A) = \emptyset$. If in addition $A^2$ is normal, $AA^* = A^*A$ since $A^2A^* = A^*A^2$. But this implies that $AA^* = A^*A$ since $\sigma(A^*) \cap (-A^*) = \emptyset$. More generally, we have by an analogous proof:

**Theorem 4.** Let $X$ be a Hilbert space. If $A$ is an element of $\mathcal{L}(X)$ such that $A^n$ is normal for some $n$ greater than 1 and $\sigma(A) \cap \sigma(e^{2\pi ik/n}A) = \emptyset$ for $k = 1, \ldots, n-1$, then $A$ is normal.

**Reference**


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