SOME COMMENTS ON THE STRUCTURE OF COMPACT DECOMPOSITIONS OF $S^3$

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In this note we derive some theorems of a general nature on compact, upper semicontinuous decompositions of $S^3$, spherical 3-space. A compact decomposition $G$ of $S^3$ is one obtained from a compact, proper subset $D$ of $S^3$ by setting $G = \{ g: g$ is either a component of $D$ or a point of $S^3 - D \}$. In this paper a point-like, compact decomposition of $S^3$ is one such that the complement of each component of $D$ is homeomorphic to $E^3$, Euclidean 3-space; and a 1-dimensional, compact decomposition of $S^3$ is one such that each component of $D$ has dimension no greater than 1.

Theorem 1 shows that in the decomposition space of a point-like, compact decomposition of $S^3$ the collection of points which fail to have some neighborhood homeomorphic to $E^3$ is dense in itself. The remaining theorems are concerned with the effect of inserting or removing certain elements from a compact decomposition of $S^3$. Armentrout's [2, Theorems 1, 6] are essential to our investigation. A version of these theorems appears below as Theorem A. A summary of some known results on compact decompositions of $S^3$ may be found in [3].

We assume then that $G$ denotes a compact decomposition of $S^3$ with associated projection map $P$ onto the decomposition space $S^3/G$. Let $H_G$ denote the sum of the nondegenerate elements of $G$. Note that $P(\text{Cl } H_G)$ is compact and 0-dimensional. It is known that there is a sequence of compact polyhedral 3-manifolds with boundary in $S^3$ such that $\bigcap_{i=1}^\infty M_i = \text{Cl } H_G$ and, for each $i$, $M_{i+1} \subseteq \text{Int } M_i$. Such a sequence $M_i$ will be referred to as a defining sequence for $G$. Let $Q = \{ x: x \in S^3/G$ and $x$ has a neighborhood homeomorphic to $E^3 \}$, and let $F = S^3/G - Q$. Note that if $F \neq \emptyset$, then $F$ is compact and 0-dimensional.

**Theorem A.** Assume $G$ is a compact decomposition of $S^3$ ($P(\text{Cl } H_G)$ is compact and 0-dimensional). Let $M$ be a compact polyhedral 3-manifold with boundary in $S^3$ such that $\text{Bd } M \cap \text{Cl } H_G = \emptyset$. If $P(M) \subseteq Q$ and either $G$ is a point-like decomposition or $M$ has a triangulation whose 1-skeleton is disjoint from $\text{Cl } H_G$, then there is a map of $M$ onto itself fixed on $\text{Bd } M$ and inducing the same decomposition as $G$ restricted to $M$.

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Proof. Since either $G$ is a point-like decomposition or $M$ has a triangulation whose 1-skeleton is disjoint from $\text{Cl} \ H_G$, it follows that there is a collection of disjoint polyhedral arcs $A_1, \ldots, A_n$ such that each $\text{Int} \ A_i \subset \text{Int} \ M - \text{Cl} \ H_G$, each $\text{Bd} \ A_i \subset \text{Bd} \ M$, and any two boundary components of any component of $M$ are connected by one of the $A_i$. For each $i$, let $A'_i$ be an open regular neighborhood of $A_i$ in $M$ such that $\text{Cl} \ A_i \cap \text{Cl} \ A'_j = \emptyset$ for $i \neq j$ and each $\text{Cl} \ A'_i \cap \text{Cl} \ H_G = \emptyset$. Let $M' = M - \bigcup_{i=1}^n A'_i$. Since the boundary of each component of $M'$ is connected, it follows from the proof of either [2, Theorem 1] (if $G$ is point-like) or [2, Theorem 6] (if $M$ has a triangulation whose 1-skeleton is disjoint from $\text{Cl} \ H_G$) that there is a homeomorphism $h'$ of $M'$ onto $P(M')$ such that $h' | \text{Bd} \ M' = P | \text{Bd} \ M'$. Extending $h'$ by using $P$ on each $A'_i$, we see that there is a homeomorphism $h$ of $M$ onto $P(M)$ such that $h | \text{Bd} \ M = P | \text{Bd} \ M$. The required map is $h^{-1}P$.

Theorem 1. Let $G$ be a point-like, compact decomposition of $S^3$. Then $F$ is either empty or a Cantor set, that is $F$ has no isolated points.

Proof. Suppose that $F$ has an isolated point $x$. In the defining sequence $M_i$ of $G$, let $n$ be chosen large enough that the component $K$ of $M_n$ containing $P^{-1}(x)$ is such that $P(K) - x \subset Q$. Since $G$ is point-like there is a map $f'$ of $K$ onto itself such that $f'$ is fixed on $\text{Bd} \ K$, $f'$ is a homeomorphism on $K - P^{-1}(x)$ and $f'(P^{-1}(x))$ is a point. For $i = 1, 2, 3, \ldots$, let $K_i = M_{n+i} \cap K$ and let $L_i$ be the component of $K_i$ containing $P^{-1}(x)$. Let $K'_i = K_i - L_i$, and, for each $i$, let $K_{i+1} = K_{i+1} - \bigcup_{j=1}^i (K'_j) \cup L_{i+1}$. Since each $P(K'_i) \subset Q$, it follows from Theorem A that there is a mapping $f_i$ of $K'_i$ onto itself fixed on $\text{Bd} \ K'_i$ and inducing the same decomposition as $G$ restricted to $K'_i$. Assuming each $f_i = \text{identity}$ on $K - K'_i$, we have that $f = f_i(\prod_{i=1}^n f_i)$ is a map of $K$ onto itself fixed on $\text{Bd} \ K$ and inducing the same decomposition as $G$ restricted to $K$. It follows that the map $Pf^{-1}$ is a homeomorphism of $K$ onto $P(K)$. But this contradicts that $x \in F$. Therefore $F$ is either empty or it is homeomorphic to a Cantor set.

In [6], Finney showed that if $f$ is a point-like, simplicial map of $S^3$, then $f(S^3)$ is homeomorphic to $S^3$. To prove this he simplified the decomposition $G_f$ of $S^3$ induced by $f$. This simplification was accomplished by a process of deleting portions of certain nondegenerate elements of $G_f$ to obtain a new decomposition $G_f'$ such that $S^3/G_f'$ is homeomorphic to $S^3/G_f$. In Theorems 2 and 3 it is shown that, for compact decompositions $G$ of $S^3$, we may delete certain elements of $G$ to form a new decomposition $G'$ such that $S^3/G$ is homeomorphic to $S^3/G'$. 

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Theorem 2. Let $G$ be a point-like, compact decomposition of $S^3$. Let $U$ be an open set in $S^3$ such that $U$ is the union of elements in $G$ and $P(U) \subseteq Q$. Let $G'$ be the decomposition obtained by points of $U$ and elements of $G$ in $S^3 - U$. Then $S^3/G$ is homeomorphic to $S^3/G'$.

Proof. It is easily checked that $G'$ is an upper semicontinuous decomposition of $S^3$. Let $P'$ be the projection map for $G'$. Let $M_i$ be a defining sequence for $G$. Define $K_i$ to be the union of all components of $M_i$ which do not intersect $S^3 - U$. Let $K_i' = K_1$ and, for $i = 1, 2, \ldots$, let $K_i' = K_{i+1} - \bigcup_{j=1}^{i} K_j'$. Since $P(K_i') \subseteq Q$ it follows by Theorem A that there is a map $f_i$ of $S^3$ onto itself fixed on $S^3 - \text{Int} K_i'$ and inducing the same decomposition as $G$ restricted to $K_i'$. Let $f = \prod_{i=1}^\infty f_i$. Then $P'fP^{-1}$ is a 1-1 correspondence between the points of $S^3/G$ and $S^3/G'$. If $K$ is a component of some $K_i'$, then $f^{-1}(K) = K$. Using this fact and that $f$ is a continuous map on $U$ that induces the same decomposition as $G$ restricted to $U$, it follows that if $V$ is open in $S^3$ and $V$ is the union of elements in $G'$, then $f^{-1}(V)$ is open in $S^3$ and $P(f^{-1}(V))$ is open in $S^3/G$. Hence $P'fP^{-1}$ is a continuous 1-1 map of $S^3/G$ onto $S^3/G'$, and it follows that $P'fP^{-1}$ is a homeomorphism between these spaces.

Theorem 3 is similar to Theorem 2, except that we consider 1-dimensional, compact decompositions of $S^3$ instead of point-like, compact decompositions of $S^3$. To prove Theorem 3 the following lemma is needed. A version of this lemma was proved independently by Alford and Sher in [1], but our proof differs from theirs in that we do not use Kwun and Raymond’s [7, Theorem 3, Corollary 2].

Lemma. Let $G$ be a 1-dimensional, compact decomposition of $S^3$ and let $M$ be a compact polyhedral 3-manifold with boundary in $S^3$ such that $\text{Bd} M \cap \text{Cl} H_G = \emptyset$, $P(M) \subseteq Q$, and $P(M)$ is embeddable in $S^3$. Then there is a map of $M$ onto itself fixed on $\text{Bd} M$ and inducing the same decomposition as $G$ restricted to $M$.

Proof. Let $T$ be a triangulation of $M$ such that each 1-simplex of $T$ that intersects $\text{Bd} M$ is disjoint from $\text{Cl} H_G \cap M$. Since $\dim g \leq 1$ for each $g \subseteq G$, we may assume that the 0-skeleton of $T$ is disjoint from $\text{Cl} H_G$. Let $A$ be a 1-simplex of $T$ with endpoints $a$ and $b$ such that $A \cap \text{Bd} M = \emptyset$, and let $K$ be a polyhedral cube in $\text{Int} M$ obtained by thickening $\text{Int} A$ slightly so that $\text{Bd} A \subseteq \text{Bd} K$ and $K \cap T_1 = A$, where $T_1$ is the carrier of the 1-skeleton of $T$. Let $U$ be a complementary domain of $\text{Bd} K$. Since $\dim g \leq 1$ for each $g \subseteq G$ and $P(\text{Cl} H_G)$ is 0-dimensional, it follows that $U - \text{Cl} H_G$ is connected. Hence there is an arc $X$ with endpoints $a$ and $b$ such that $\text{Int} X \subseteq U$ and $X \cap \text{Cl} H_G$.
Similarly there is an arc \( Y \) with endpoints \( a \) and \( b \) such that \( \text{Int } Y \subset (S^3 - \text{Cl } U) \) and \( Y \cap \text{Cl } H_a = \emptyset \). Since \( \text{Bd } M \cap \text{Cl } H_a = \emptyset \), we may assume \( X \cup Y \subset \text{Int } M \).

Suppose \( \text{Bd } K \cap \text{Cl } H_a \) separates \( a \) from \( b \) on \( \text{Bd } K \). Let \( S \) be the component of \( \text{Bd } K - \text{Cl } H_a \) containing \( a \). It follows that \( P(\text{Cl } S) \) is a singular 2-sphere in \( \text{Int } P(M) \) (regard \( P(\text{Cl } S) \) as the image of \( \text{Bd } K \) under the map \( g(x) = P(x) \) if \( x \in S \) and \( g(x) = P(\text{Bd } C) \) if \( x \) belongs to the component \( C \) of \( \text{Bd } K - S \)) and that the simple closed curve \( P(X \cup Y) \) intersects and pierces \( P(\text{Cl } S) \) at just one point. But this is impossible since \( P(M) \) is embeddable in \( S^3 \). Hence there is an arc \( B \) on \( \text{Bd } K \) with endpoints \( a \) and \( b \) such that \( B \cap \text{Cl } H_a = \emptyset \). There is a homeomorphism of \( M \) onto itself which is fixed on \((\text{Bd } M \cup T_1) - \text{Int } A\) and takes \( A \) onto \( B \). Repeating this argument a finite number of times, we push \( T_1 \) off \( \text{Cl } H_G \). Since \( P(M) \subset Q \), this lemma now follows from Theorem A.

**Theorem 3.** Let \( G \) be a 1-dimensional, compact decomposition of \( S^3 \) and let \( U \) be an open set in \( S^3 \) such that \( U \) is the union of elements in \( G \) and \( P(U) \subset Q \). Let \( G' \) be the decomposition obtained by points of \( U \) and elements of \( G \) in \( S^3 - U \). Then \( S^3/G \) is homeomorphic to \( S^3/G' \).

**Proof.** As in Theorem 2 we may choose a sequence \( K'_1 \) of disjoint 3-manifolds in \( U \) such that \( U \cap \text{Cl } H_a \subset \bigcup_{i=1}^{n} K'_i \). Each \( K'_i \) can be chosen so that \( P(K'_i) \) is embeddable in \( S^3 \). By the previous lemma there is a map of \( S^3 \) onto itself fixed on \( S^3 - \text{Int } K'_i \) and inducing the same decomposition as \( G \) restricted to \( K'_i \). The remainder of this proof is the same as the proof of Theorem 2.

**Corollary.** Let \( G \) be a 1-dimensional, compact decomposition of \( S^3 \) such that \( S^3/G \) is a 3-manifold. Then \( S^3/G \) is homeomorphic to \( S^3 \).

As pointed out by Bing in [4, p. 7], it may be shown that if \( C_1, C_2, \cdots, C_n \) are mutually exclusive, nonseparating continua in \( S^3 \), then there is a decomposition \( G \) of \( S^3 \) such that each \( C_i \in G \) and \( S^3/G \) is homeomorphic to \( S^3 \). The next theorem shows that while staying in the category of point-like, compact decompositions we cannot change an element of \( G \) whose image is in \( F \) to one whose image is in \( Q \) by adding more nondegenerate elements to \( G \).

**Theorem 4.** Let \( G \) and \( G' \) be point-like, compact decompositions of \( S^3 \) such that each nondegenerate element of \( G \) is contained in \( G' \). If \( g \in G \) and \( P(g) \in F \), then \( P'(g) \in F' \) (where \( P' \) is the projection map associated with \( G' \) and \( F' \) is the non-Euclidean points of \( S^3/G' \)).
Proof. Assume by way of contradiction that $g_0$ is an element of $G$ such that $P(g_0) \subseteq F$ but $P'(g_0) \subseteq Q'$ (where $Q' = S^3/G' - F'$). Let $M_i$ and $M'_i$ be defining sequences for $G$ and $G'$, respectively. Let $K$ be a component of some $M_n$ such that $g_0 \subseteq K$, $P'(K) \subseteq Q'$, and, for each $i = 1, 2, \ldots$, let $K_i = K \cap M_{n+i}$. If for each positive integer $m$ and each positive number $\epsilon$ there is a homeomorphism $h$ of $S^3$ onto itself such that: (1) if $x \in S^3 - K_m$, then $h(x) = x$; and (2) if $g \in G$ and $g \subseteq K_m$, then $\text{diam } h(g) < \epsilon$, then it follows from the proof of Theorem 1 of [5] that $P(K)$ is homeomorphic to $K$. This would contradict that $P(g_0) \subseteq F$ and establish the theorem.

Hence let $m$ and $\epsilon$ be given. Let $K'$ be the union of all elements of $G$ contained in $K$ and of diameter greater than or equal to $\epsilon$. There is a positive integer $p$ such that the components of $M'_p$ which intersect $K'$ are contained in $K_m$. Denote the union of these components by $S$. Since $P'(S) \subseteq Q'$, it follows from the proof of Theorem 2 of [2] that there is a homeomorphism $h$ of $S^3$ onto itself such that: (1) if $x \in S^3 - S$, then $h(x) = x$; and (2) if $g' \in G'$ and $g' \subseteq S$, then $\text{diam } h(g') < \epsilon$. This homeomorphism satisfies the desired properties given in the previous paragraph and completes the proof.

References