A PROBLEM ON PARTITIONS CONNECTED WITH WARING'S PROBLEM

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1. **Introduction.** Let k, s be fixed positive integers, and n an arbitrary positive integer. Then we denote by $R(n)$ the number of representations of n as a sum of s kth powers of positive integers; that is, $R(n)$ is the number of solutions $(x_1, x_2, \cdots, x_s)$ of the Diophantine equation

$$(1) \qquad n = x_1^k + x_2^k + \cdots + x_s^k \qquad (x_i \text{ positive integers}),$$

solutions differing only in the order of the x_i being counted as distinct.

Hardy and Littlewood discovered the famous asymptotic formula

$$(2) \qquad R(n) = \frac{\Gamma^s(1 + 1/k)}{\Gamma(s/k)} \mathfrak{S}(n) n^{s/k-1} + o(n^{s/k-1}) \qquad (n \to \infty),$$

where $\mathfrak{S}(n)$ is the 'singular series', and Hua [**3**] proved that (2) holds for $s \geqq 2^k+1$. An elegant and short proof of Hua's theorem was published, in 1948, by Estermann [**2**]. A more powerful method, however, was developed by Vinogradov, who showed that (2) holds for $s \geqq [10k^2 \log k]$ provided $k \geqq 12$ (see [**7**, Chapter VII]).

We have reckoned the number $R(n)$ considering the order of the x_i. If, however, we count the number of solutions of (1) without regard to the order of the summands, we get a problem of partitions. This problem seems to be open except for $k=1$. When $k=1$, on the other hand, there is a considerable literature on the problem (see H. Ostmann [**5**, p. 52], G. J. Rieger [**6**]).

The main purpose of the present paper is to establish the following theorem.

THEOREM 1. *Let $P(n)$ denote the number of partitions of a positive integer n into s kth powers of positive integers. Then, for $s \geqq 2^k+1$ $(k \geqq 2)$ or $s \geqq [10k^2 \log k]$ $(k \geqq 12)$, we have*

$$(3) \qquad P(n) = \frac{\Gamma^s(1 + 1/k)}{s!\Gamma(s/k)} \mathfrak{S}(n) n^{s/k-1} + o(n^{s/k-1}) \qquad (n \to \infty).$$

Comparing (3) with (2), it is observed that the only difference of

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the main term of $P(n)$ from that of $R(n)$ is $s!$ in the denominator. It may also be noted that our conditions on s for the validity of (3) are identical with those of Hua and of Vinogradov mentioned above.

2. Henceforth we assume that $k \geqq 2$ and $s \geqq 2$. First, we define $R_1(n)$ as the number of solutions of (1) in which $x_1, x_2, \cdots, x_s$ are distinct, and $R_2(n)$ as the number of solutions in which at least two of the x_i are equal, the order of the x_i being relevant in each case. Then clearly

$$R(n) = R_1(n) + R_2(n). \tag{4}$$

Secondly, we regard (1) as a partition of n, and, corresponding to the above, define $P_1(n)$ as the number of partitions in which $x_1, x_2, \cdots, x_s$ are distinct, and $P_2(n)$ as the number of partitions in which at least two of the x_i are equal, the order of the x_i being, of course, irrelevant. Then we have also

$$P(n) = P_1(n) + P_2(n). \tag{5}$$

Moreover, it easily follows that

$$R_1(n) = s!P_1(n), \tag{6}$$

$$P_2(n) \leqq R_2(n) \leqq s!\, P_2(n)/2! \tag{7}$$

Suppose now that (2) holds for some s and further that

$$R_2(n) = o(n^{s/k-1}). \tag{8}$$

Then we have, by (4),

$$R_1(n) = \frac{\Gamma^s(1 + 1/k)}{\Gamma(s/k)} \mathfrak{S}(n) n^{s/k-1} + o(n^{s/k-1}), \tag{9}$$

and, by (7),

$$P_2(n) = o(n^{s/k-1}). \tag{10}$$

Therefore, we infer from (5), (6), (9), and (10)

$$P(n) = \frac{\Gamma^s(1 + 1/k)}{s!\Gamma(s/k)} \mathfrak{S}(n) n^{s/k-1} + o(n^{s/k-1}), \tag{3}$$

that is, (3) follows from (2) and (8). Conversely, we can show that (8) follows from (2) and (3). Indeed, we obtain, from (4), (5), (6), and (7),

$$\begin{aligned} s!P(n) - R(n) &= s!P_1(n) + s!P_2(n) - R_1(n) - R_2(n) \\ &= s!P_2(n) - R_2(n) \geqq 2R_2(n) - R_2(n) = R_2(n). \end{aligned}$$

The left-hand side of this inequality is $o(n^{s/k-1})$ by (2), (3); and hence (8) follows. Consequently we have the following lemma.

LEMMA 1. (3) *and* (8) *are equivalent expressions for those values of* s *for which* (2) *is valid.*

3. It would be difficult, however, to calculate $R_2(n)$ precisely, and so we employ the following method:

If $Q(n)$ denotes the number of solutions of (1) (considering the order of the summands) in which $x_1 = x_2$ holds, then obviously

$$(11)\qquad Q(n) = \int_{\alpha_0}^{\alpha_0+1} T^{s-2}(\alpha) T_1(2\alpha) e(-n\alpha)\, d\alpha \qquad (\alpha_0 \text{ any real number}),$$

where

$$T(\alpha) = \sum_{x=1}^{P} e(\alpha x^k), \qquad P = [n^{1/k}],$$

$$T_1(\alpha) = \sum_{x=1}^{P_1} e(\alpha x^k), \quad P_1 = [(n/2)^{1/k}]; \quad e(z) = e^{2\pi i z}.$$

More generally, it will be seen easily that $Q(n)$ equals the number of solutions of (1) in which $x_i = x_j$ for any fixed numbers i, j $(i \neq j)$ holds. Since there are $s!/2!(s-2)!$ such pairs (i, j) taken from $1, 2, \cdots, s$, we obtain

$$(12)\qquad Q(n) \leqq R_2(n) \leqq \binom{s}{2} Q(n).$$

From (12) it follows that (8) is equivalent to

$$Q(n) = o(n^{s/k-1}).$$

4. The number $Q(n)$ can be treated by analytic methods similar to those developed for Waring's Problem. In the first place, we shall follow the pattern of Estermann's version [2] of Hua's paper [3]; next we adopt Vinogradov's method to obtain a sharper result for large k.

Let a, q be any pair of integers such that $1 \leqq a \leqq q$, $(a, q) = 1$. We write $I(a, q)$ for the interval $(a-\alpha_0)/q \leqq \alpha \leqq (a+\alpha_0)/q$ where $0 < \alpha_0 < \frac{1}{2}$. Let ν be a real number satisfying

$$(13)\qquad 1 < \nu < (2\alpha_0)^{-1}.$$

Then it will be verified by a slight calculation that the intervals $I(a, q)$ with $q \leqq \nu$ are nonoverlapping, and hence, by (11),

$$Q(n) = \sum_{1\leq q\leq \nu} \sum_{a} J(a, q) + \int_E T^{s-2}(\alpha)T_1(2\alpha)e(-n\alpha)d\alpha$$
$$= Q^*(n) + Q^{**}(n),$$

say, where

$$J(a, q) = \int_{I(a,q)} T^{s-2}(\alpha)T_1(2\alpha)e(-n\alpha)d\alpha,$$

and E is the set of those numbers of the interval $\alpha_0 \leqq \alpha < 1$ which do not belong to any $I(a, q)$ with $q \leqq \nu$.

Assume now that we have an estimate for $T(\alpha)$ such that

(14) $$T(\alpha) \ll P^{1-\rho} \qquad (\alpha \in E, \rho = \rho(k) > 0)$$

and also that

(15) $$\int_0^1 | T(\alpha) |^t d\alpha \ll P^{t-k+\delta} \qquad (\delta = \delta(k) > 0)$$

where $t = t(k)$ is some positive integer. Then we obtain, for $s \geqq t+1$,

$$Q^{**}(n) \ll P^{(s-t-1)(1-\rho)} \int_0^1 | T(\alpha) |^{t-1} | T_1(2\alpha) | d\alpha.$$

Here, by Hölder's inequality (noting the periodicity of $T_1(\alpha)$),

$$\int_0^1 | T(\alpha) |^{t-1} | T_1(2\alpha) | d\alpha$$
$$\leqq \left(\int_0^1 | T(\alpha) |^t d\alpha\right)^{1-1/t} \left(\int_0^1 | T_1(\alpha) |^t d\alpha\right)^{1/t},$$

and the right member is, by (15), $\ll P^{t-k+\delta}$. Hence we get the following estimate:

(16) $$Q^{**}(n) \ll P^{s-k-\mu},$$

where

(17) $$\mu = 1 - \delta + (s - t - 1)\rho.$$

If we can prove that $\mu > 0$ for $s \geqq s_1(k)$, it then follows from (16) that

(18) $$Q^{**}(n) = o(n^{s/k-1}),$$

provided $s \geqq s_1(k)$.

Now let us first put $\nu = n^{1/(4k)}$ and $\alpha_0 = \nu/n$. Then (13) is fulfilled whenever $n \geqq 3$. (14) and (15) are also valid with $\rho = 2^{-k-1} - \epsilon$, $t = 2^{k-1}$, and $\delta = 1 + \epsilon$ (see [2, Lemmas 7, 4($m = k-1$)]), where ϵ is an arbitrarily small positive number. Hence when $s \geqq s_1(k) = 2^{k-1}+2$, we have, by (17),

$$\mu = -\epsilon + (s - 2^{k-1} - 1)(2^{-k-1} - \epsilon) \geqq 2^{-k-1} - 2\epsilon > 0,$$

and therefore (18) holds. We now discuss $Q^*(n)$. It will be seen that a crude estimate for $Q^*(n)$ is sufficient for our purpose. Using the trivial inequalities: $|T(\alpha)| \leqq P$, $|T_1(2\alpha)| \leqq P_1 < P$, we find that

$$|J(a, q)| < \int_{I(a,q)} P^{s-1} d\alpha = 2\alpha_0 q^{-1} P^{s-1} = 2\nu n^{-1} q^{-1} P^{s-1},$$

from which it follows that

$$\left| \sum_{1 \leqq q \leqq \nu} \sum_{a} J(a, q) \right| < 2\nu n^{-1} P^{s-1} \sum_{1 \leqq q \leqq \nu} \sum_{1 \leqq a \leqq q} q^{-1} \leqq 2\nu^2 n^{-1} P^{s-1}$$
$$\leqq 2n^{2/(4k)+(s-1)/k-1} = 2n^{s/k-1-1/(2k)}.$$

Thus, $Q^*(n) = o(n^{s/k-1})$, and so finally $Q(n) = o(n^{s/k-1})$ provided $s \geqq 2^{k-1}+2$.

We next turn to Vinogradov's treatment to obtain a better result for large k. We put $\nu = P^{1-1/k}$, $\alpha_0 = (2k)^{-1} P^{1-k}$. These values again satisfy (13). By virtue of Vinogradov's results [**7**, Chapter VII], we see that both (14) and (15) hold with $\rho = (3k(k-1)\log(12k^2))^{-1}$, $t = 2b(m+h)$, and $\delta = \frac{1}{2}k(k+1)\sigma$, where $k \geqq 12$, $b = [\frac{5}{4}k + \frac{1}{2}]$, $h = k+2$, $\sigma = (1-1/k)^m$, and m is any fixed integer greater than k. Let us now take

$$m = \left[\frac{\log(0.5k(k+1))}{-\log(1 - 1/k)} + 1 \right],$$

which ensures that $\sigma < (0.5k(k+1))^{-1}$, whence we get $\delta < 1$. If $s \geqq t+2 = 2b(m+h)+2$, we have therefore $\mu > (s-t-1)\rho \geqq \rho > 0$. Now a simple calculation shows that

$$2b(m + h) < 5k^2 \log k + 2.5(1 - \log 2)k^2 + 11k + 3$$
$$< 6k^2 \log k - 2 \qquad (k \geqq 12).$$

Hence if $s \geqq s_1(k) = [6k^2 \log k]$, we obtain $s > 6k^2 \log k - 1 > 2b(m+h) + 1$, so that $s \geqq 2b(m+h)+2$; and thus (18) holds. There is no difficulty in dealing with $Q^*(n)$ if we utilize an analysis analogous to that given in [**7**, Chapter III] (cf. Davenport [**1**, pp. 50–51]); indeed we can deduce that

$$Q^*(n) = O(P^{s-1-k}) = o(n^{s/k-1}).$$

Consequently we have $Q(n)=o(n^{s/k-1})$, provided that $s\geqq[6k^2\log k]$ $(k\geqq 12)$.

The above arguments, together with (12), yield the following theorem.

THEOREM 2. *Let $R_2(n)$ denote the number of representations of n as a sum of s kth powers of positive integers where not all of the summands are distinct. Then, for $s\geqq 2^{k-1}+2$ $(k\geqq 2)$ or $s\geqq[6k^2\log k]$ $(k\geqq 12)$, we have*

$$R_2(n) = o(n^{s/k-1}). \tag{8}$$

In particular, if $s\geqq 2^k+1$, then (8) is valid since $2^k+1>2^{k-1}+2$ $(k\geqq 2)$ and also (2) holds by Hua's theorem. A similar argument applies to the case $s\geqq[10k^2\log k]$. This proves Theorem 1 on account of Lemma 1.

REMARK. It is noteworthy that the number $2^{k-1}+2$, appearing in Theorem 2, is comparatively small for small values of k (see the table below). It is interesting to see that the values of $2^{k-1}+2$ for $3\leqq k\leqq 6$ are respectively less than the best known upper bounds for $G(k)$ (i.e. $G(3)\leqq 7$, $G(4)=16$, $G(5)\leqq 23$, $G(6)\leqq 36$). As regards the case $k=2$, a slightly better result than that of Theorem 2 holds; we have, in fact,

$$R_2(n) = O(n^{(s-3)/2+\epsilon})$$

for $s\geqq 3$ (cf. Landau [**4**, Theorem 204], Evelyn and Linfoot [**8**, Lemma 2.2]).

k	2	3	4	5	6	7	8	9	10
$2^{k-1}+2$	4	6	10	18	34	66	130	258	514
2^k+1	5	9	17	33	65	129	257	513	1025

5. Professor H. Davenport has raised (private communication) the following question:

If $G_0(k)$ denotes the least value of s_0 such that the Hardy-Littlewood formula (2) holds[2] for $s\geqq s_0$, then does the formula (3) hold as well for $s\geqq G_0(k)$?

[2] In order that (2) may be an asymptotic formula for $R(n)$, it should be required that $\mathfrak{S}(n)\geqq c(k,s)>0$ for all sufficiently large n, and also we have $G_0(k)\geqq G(k)$. Thus $G_0(2)=5$, though when $k=2$ and $3\leqq s\leqq 8$, we have *exact formulae* for the number of solutions of (1) if we allow the x_i to be zero or negative integers (see [9]). The value of $G_0(k)$ is not known for $k>2$.

After Lemma 1 and (12), this problem amounts to determining whether $Q(n)=o(n^{s/k-1})$ holds for $s\geqq G_0(k)$.

The author is unable to solve this problem completely, and we shall give here a less satisfactory answer as follows:

Formula (3) *is true if* $s\geqq G_0(k)+2$.

The proof of this is easy. For we have, if $s\geqq G_0(k)+2$,

$$Q(n) = \sum_{x=1}^{P_1} R(n-2x^k, s-2) \ll \sum_{x=1}^{P_1} (n-2x^k)^{(s-2)/k-1+\epsilon}$$

$$\ll \int_0^{(n/2)^{1/k}} (n-2x^k)^{(s-2)/k-1+\epsilon} dx \ll n^{(s-1)/k-1+\epsilon},$$

and thus $Q(n)=o(n^{s/k-1})$, where $R(n, s-2)$ denotes the number of solutions of (1) with $s-2$ summands in place of s, and where we have used the fact that the singular series $\mathfrak{S}(n)$, appearing in (2), is subject to the estimate $O(n^\epsilon)$ for $s>k$. (Actually, we can prove, by using the results [**4**, VI, Chapter 2, §§2, 4], that $\mathfrak{S}(n)=O((\log\log n)^c)$ $(c=c(k)>0)$ when $s=k+1$ and $\mathfrak{S}(n)=O(1)$ when $s>k+1$, provided $k\geqq 3$.)

As $Q(n)$ is the number of solutions of

$$2x_1^k + x_3^k + \cdots + x_s^k = n,$$

which has $s-1$ variables, it is known that $Q(n)$ satisfies an analogous asymptotic formula (see [**3**], [**1**, Theorem 4]), namely

$$Q(n) = 2^{-1/k}\frac{\Gamma^{s-1}(1+1/k)}{\Gamma((s-1)/k)}\mathfrak{S}_1(n)n^{(s-1)/k-1} + o(n^{(s-1)/k-1}).$$

It seems probable that this formula is also valid for $s-1\geqq G_0(k)$, in agreement with formula (2). If this is true, we should have

$$Q(n) = O(n^{(s-1)/k-1+\epsilon}) = o(n^{s/k-1})$$

for $s\geqq G_0(k)+1$, which extends the validity of (3) to $s\geqq G_0(k)+1$.

It is quite possible[3] that $Q(n)=o(n^{s/k-1})$ holds for $s\geqq G_0(k)$ or more values of s, giving thereby an affirmative answer to our question. But this conjecture seems difficult to prove unless the actual value of $G_0(k)$ is known.

It should be referred to in this connection that Hardy and Littlewood [**10**, p. 4] had introduced the 'Hypothesis K' which asserts that $R(n, k)=O(n^\epsilon)$ for every positive ϵ. Although this hypothesis has

[3] See the remark at the end of §4.

proved false when $k=3$, it is still plausible that one has at any rate $R(n, k)=o(n^{1/k})$, which is much weaker than Hypothesis K and may be compared with $Q(n)=o(n^{s/k-1})$ where $s=k+1$. If the estimate $Q(n)=o(n^{s/k-1})$ is valid when $s=k+1$, it may be shown by an elementary argument that the same estimate holds generally for $s\geqq k+1$. We are thus led to state the following

CONJECTURE. *Let $Q(n, k)$ $(k\geqq 3)$ denote the number of solutions of*

$$2y_1^k + y_2^k + \cdots + y_k^k = n$$

in positive integers y_i. Then $Q(n, k)=o(n^{1/k})$.

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