DIFFERENTIABILITY ALMOST EVERYWHERE OF FUNCTIONS OF SEVERAL VARIABLES[1]

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1. The purpose of this paper is to extend the results of C. J. Neugebauer in [3], where necessary and sufficient conditions for a measurable function of a real variable to be equivalent to one which is differentiable are given, to the case of higher order differentiation and to functions of several real variables.

Let $f: E_n \to R$ be a measurable function from n-dimensional Euclidean space to the reals. Q will denote a cube in E_n; and if E is a measurable set in E_n, we denote its measure by $|E|$. All functions and sets will be assumed to be measurable. Suppose that we can write $f(x_0+t) = P_{x_0}(t)+R_{x_0}(t)$ where $P_{x_0}(t)$ is a polynomial of degree $\leqq k-1$, then we say f has a $k-1$ derivative at x_0 in the L^p sense $(1 \leqq p \leqq \infty)$ if

$$\left\{\frac{1}{\rho^n}\int_{|t|\leq\rho} |R_{x_0}(t)|^p dt\right\}^{1/p} = o(\rho^{k-1}), \qquad \rho \to 0. \tag{1}$$

We write

$$\overset{k}{\Delta}_{x_0}(f, t) = \overset{k}{\Delta}_{x_0}(t) = 2\overset{k}{\epsilon}_{x_0}(t)\,|t|^k = R_{x_0}(t) + (-1)^{k-1}R_{x_0}(-t).$$

If $R_{x_0}(t) = o(|t|^{k-1})$ as $|t| \to 0$, we say f has a $k-1$ derivative at x_0; and if $R_{x_0}(t) = o(|t|^{k-1})$ as $|t| \to 0$ through a set of Lebesgue density 1 at x_0, we say f has an approximate $k-1$ derivative at x_0. Finally, if $R_{x_0}(t) = o(|t|^{k-1})$ for each x_0 in a set E provided $t \to 0$ through a set whose complement has measure zero, we say f has a $(k-1)_*$ derivative for each x_0 in E.

With these remarks we can state the following known results.

THEOREM A (SEE [6]). *f has a kth derivative in the L^2 sense for almost every x_0 in E, $|E| > 0$, if and only if for almost every x_0 in E there exists an $\eta_{x_0} = \eta > 0$ such that*

$$\int_{|t|\leq\eta} \frac{|\overset{k}{\Delta}_{x_0}(t)|^2}{|t|^{n+2k}}\, dt < \infty \cdot \tag{2}$$

THEOREM B. *f has a kth derivative in the L^p sense $(2 \leqq p < \infty)$ at almost every x_0 in E, $|E| > 0$, if and only if for almost every x_0 in E there exists $\eta_{x_0} = \eta > 0$ such that*

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$$(3) \qquad \int_{|t|\leq\eta} \frac{|\Delta_{x_0}^{k}(t)|^2}{|t|^{n+2k}}\,dt < \infty, \qquad \int_{|t|\leq\eta} \frac{|\Delta_{x_0}^{k}(t)|^p}{|t|^{n+pk}}\,dt < \infty.$$

Theorem A is due to Stein and Zygmund and was originally proved in [**5**] for the case of a function of a single variable. In [**6**] it is shown how it can be extended as indicated here. Theorem B was proved in [**7**] for the case $k=1$. With Theorem A one can prove Theorem B by using a similar argument to that found in [**5**, p. 280].

The theorem we wish to prove is the following one.

THEOREM 1. *f is equivalent to a function which has a kth derivative almost everywhere in a set E, $|E|>0$, if and only if for almost every x in E there is an $\eta_x=\eta>0$ such that the sequence*

$$(4) \qquad \left\{\int_{|t|\leq\eta} \frac{|\Delta_{x_0}^{k}(t)|^p}{|t|^{n+pk}}\,dt\right\}^{1/p}, \qquad p = 2, 3, \cdots,$$

is bounded.

2. We need some definitions and lemmas to enable us to prove these theorems. Let $E_y=\{x: f(x)>y\}$ for real values of y. Set $g(x_0) =\inf_y\{y: |E_y\cap Q| =o(|Q|)$ as $|Q|\to 0$, $x_0\in Q\}$. Then g: $E_n\to R$ is a function, is called the upper boundary of f (see [**1**]), and satisfies the properties:

(P_1) $f(x)=g(x)$ at every point of approximate continuity of f, and hence f is equivalent to g;

(P_2) for x_0 in E_n and $\epsilon>0$, $\eta>0$, the set $\{x: |x-x_0|<\eta$ and $|g(x)-g(x_0)|<\epsilon\}$ has positive measure.

LEMMA 1. *Let E be a measurable set and suppose f has a $(k)_*$ derivative at each point of E. Then f is equivalent to a function which has a kth derivative at each point of E.*

PROOF. Let g be the upper boundary of f. Then f is equivalent to g and $f(x)=g(x)$ at every point of E. Let x_0 be in E and let $\epsilon>0$ be given. We may assume $P_{x_0}(t)=0$. There is a $\delta>0$ and a set N, $|N|=0$, and $N\subset\{x: |x-x_0|<\delta\}$ such that $|g(x_0+t)|\leqq\epsilon|t|^k$ provided $x_0+t\notin N$ and $|t|<\delta$. If $N\neq\varnothing$, there is a $\sigma>0$ and a point t_0, $|t_0|<\delta$, and $(x_0+t)\in N$ such that $|g(x_0+t_0)|\geqq(\epsilon+\sigma)|t_0|^k$. By property ($P_2$) there is a sequence $\{t_n\}$, $t_n\to t_0$ and x_0+t_n is not in N, such that $g(x_0+t_n)\to g(x_0+t_0)$. This implies $|g(x_0+t_0)|\leqq\epsilon|t_0|^k$ from which we conclude $N=\varnothing$, and the lemma follows.

LEMMA 2. *Assume that for $x\in E$, $|E|>0$, there is a set N_x, $|N_x|=0$, such that $\Delta_x^k(t)=O(|t|^k)$ as $|t|\to 0$ for t in N_x. Then there exists a set N,*

$|N|=0$, *such that* $\Delta_x^k(t)=O(|t|^k)$ *as* $|t|\to 0$ *with* $(x\mp t)\notin N$ *for each* x *in* E.

PROOF. Let N be the set of points at which f is not approximately continuous. $|N|=0$, and it can easily be shown that $\Delta_x^k(t)=O(|t|^k)$ as $|t|\to 0$ with $(x\mp t)\notin N$ for each x in E.

LEMMA 3. *Let* $f: E_n\to R$ *be measurable and* $1\leqq p\leqq\infty$. *Then at all points* x, *at which* f *has a kth derivative in the* L^p *sense,* f *has an approximate kth derivative.*

LEMMA 4. *Let* $f: E_n\to R$ *be measurable such that at each* x_0 *in a set* E, $|E|>0$, f *has a kth approximate derivative and that the polynomial* $P_{x_0}(t)$ *is zero. Suppose that for each* x_0 *in* E *there is a set* N_{x_0} *such that* $|N_{x_0}|=0$ *and* $\Delta_{x_0}^k(t)=O(|t|^k)$ *as* $|t|\to 0$, $t\notin N_{x_0}$. *Then* f *has a* $(k)_*$ *derivative at almost all* $x_0\in E$.

PROOF. As in Lemma 2, there is a set N, $|N|=0$, such that $\Delta_x^k(t)=O(|t|^k)$ as $|t|\to 0$ with $x\pm t\notin N$ for each $x\in E$. Let $F_1\subset F_2\subset\cdots$ be a sequence of closed sets such that f restricted to F_i is continuous, $F_i\subset E_n-N$, and $H=\bigcup_{i=1}^{\infty} F_i$ has full measure in every cube. Let

$$E(m,j)=\{x\in E\cap H: |\Delta_x^k(t)| < m|t|^k, 0<|t|<1/j, x+t\in H\}.$$

One can show these sets are measurable (see e.g. [**4**]). We will show that at each point of density of $E(m,j)$ f has a $(k)_*$ derivative, and this will complete the proof.

Assume x_0 is a point of density of $E(m,j)$. Let $\sigma>0$ be given. Then there is a set $E_1\subset E(m,j)$ with x_0 as a point of density, and if x_0+t is in E_1 we have $|f(x_0+t)|\leqq\sigma|t|^k$. Let

$$A_t=\{x:(x_0+x+t)/2\in E_1,\ 0<|x-x_0|<|t| \quad \text{and} \quad |x_0+t-x|<(\sigma/m)^{1/k}|t|\}.$$

Let χ be the characteristic function of E_1, then A_t has measure equal to

$$\int_{|x_0+t-x|<(\sigma/m)^{1/k}|t|}\chi\left(\frac{x_0+x+t}{2}\right)dx.$$

Since x_0 is a point of density of the set E_1, this integral is asymptotically equal to $C_n(\sigma/m)^{n/k}|t|^n$ as $|t|\to 0$ where C_n is a constant depending only on n. Hence there is $0<\delta<1/j$ such that $A_t\cap E\neq\varnothing$ whenever $0<|t|<\delta$. Thus for every $|t|$, $0<|t|<\delta$, there is an $x\in A_t\cap E_1$. Since $x\in E_1$ we have $|f(x)|\leqq\sigma|x-x_0|^k\leqq\sigma|t|^k$ and if x_0+t is in H we obtain

$$\begin{aligned} |f(x_0+t)| &\leqq |f(x_0+t)+(-1)^{k-1}f(x)| + |f(x)| \\ &\leqq m|(x_0+t-x)/2|^k + \sigma|t|^k \\ &\leqq m(\sigma/m)|t|^k + \sigma|t|^k = 2\sigma|t|^k. \end{aligned}$$

Hence f has a kth derivative at x_0 relative to the set H.

LEMMA 5 (SEE [2, p. 18]). *Let f be a real valued function in $L^p(E_n)$ $(1 \leqq p \leqq \infty)$ and suppose f has a kth derivative in the L^p sense at each point of a set E of positive measure. Then, for $\epsilon > 0$, there is a closed set S, $|E-S| < \epsilon$, and two functions f_1 and f_2 such that $f = f_1 + f_2$ with $f_1 \in C^k$, and for x_0 in S,*

$$\int_{|t| \leqq \rho} |f_2(x_0+t)|^p dt = o(\rho^{pk+n}), \qquad \rho \to 0.$$

3. We are now ready to give the proof of Theorem 1. First let us assume that there is a function g equivalent to f which has a kth derivative almost everywhere in E. By Theorem A, for almost every x in E, there is an $\eta_x = \eta > 0$ such that

$$\int_{|t| \leqq \eta} \frac{|\overset{k}{\epsilon}_x(t)|^2}{|t|^n} dt < \infty. \tag{5}$$

We will show that the sequence (4) is bounded at every point x for which $f(x) = g(x)$, g has a kth derivative and (5) holds. Since g has a kth derivative, $\Delta_x^k(g, t) = o(|t|^k)$. Also, since $f(x) = g(x)$ almost everywhere, there is an η', $0 < \eta' < \eta$, and a set N_x of zero measure such that $\Delta_x^k(t) < |t|^k$ for $|t| < \eta'$ and $t \notin N_x$. Thus for $p \geqq 2$

$$\int_{|t| \leqq \eta'} \frac{|\overset{k}{\epsilon}_x(t)|^p}{|t|^n} dt \leqq 2^{2-p} \int_{|t| \leqq \eta'} \frac{|\overset{k}{\epsilon}_x(t)|^2}{|t|^n} dt.$$

Hence, (4) is bounded with η replaced by η'.

Now assume (4) is bounded, with $M < \infty$ as bound. Then we have ess sup$|\Delta_{x_0}^k(t)| \leqq M|t|^k$, $0 < |t| < \eta$. The finiteness of the first term of the sequence $(p=2)$ implies the existence of the kth derivative in the L^2 sense almost everywhere in E and that f is in L^2 of a neighborhood of almost every point of E. Let $\epsilon > 0$ be given. By Lemma 5, there is a set S such that $f = f_1 + f_2$, $|E-S| < \epsilon$, f_1 is C^k, and for x in S,

$$\int_{|t| \leqq \rho} |f_2(x+t)|^2 dt = o(\rho^{2k+n}). \tag{6}$$

Since ϵ is arbitrary, it is enough to show that f_2 is equivalent to a function which has k derivatives at almost every point of S. Equation (6) and Lemma 3 imply that f_2 has a kth approximate derivative at the points x_0 in S and that the polynomial $P_{x_0}(t)$ for f_2 is zero. Since f_1 is C^k, it follows that $\Delta_x^k(f_1, t)=o(|t|^k)$ and hence $\Delta_x^k(f_2, t)=O(|t|^k)$ provided $|t|<\eta$ and $t\notin N_x$, $|N_x|=0$. Now apply Lemma 4 and Lemma 1.

We point out that the boundedness of (4) and Theorem B imply that f has a kth derivative in the L^p sense $(2\leqq p<\infty)$ almost everywhere in E. The boundedness of (4) may be regarded as a condition for the existence of the kth derivative in the L^p sense $(2\leqq p<\infty)$ almost everywhere in E and uniformly in p. This is precisely what is needed to pass to the $p=\infty$ case, as was pointed out in [3] for a function of a single variable.

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