

DIFFERENTIABILITY ALMOST EVERYWHERE OF FUNCTIONS OF SEVERAL VARIABLES¹

G. V. WELLAND²

1. The purpose of this paper is to extend the results of C. J. Neugebauer in [3], where necessary and sufficient conditions for a measurable function of a real variable to be equivalent to one which is differentiable are given, to the case of higher order differentiation and to functions of several real variables.

Let $f: E_n \rightarrow R$ be a measurable function from n -dimensional Euclidean space to the reals. Q will denote a cube in E_n ; and if E is a measurable set in E_n , we denote its measure by $|E|$. All functions and sets will be assumed to be measurable. Suppose that we can write $f(x_0+t) = P_{x_0}(t) + R_{x_0}(t)$ where $P_{x_0}(t)$ is a polynomial of degree $\leq k-1$, then we say f has a $k-1$ derivative at x_0 in the L^p sense ($1 \leq p \leq \infty$) if

$$(1) \quad \left\{ \frac{1}{\rho^n} \int_{|t| \leq \rho} |R_{x_0}(t)|^p dt \right\}^{1/p} = o(\rho^{k-1}), \quad \rho \rightarrow 0.$$

We write

$$\Delta_{x_0}^k(f, t) = \Delta_{x_0}^k(t) = 2\epsilon_{x_0}^k(t) |t|^k = R_{x_0}(t) + (-1)^{k-1} R_{x_0}(-t).$$

If $R_{x_0}(t) = o(|t|^{k-1})$ as $|t| \rightarrow 0$, we say f has a $k-1$ derivative at x_0 ; and if $R_{x_0}(t) = o(|t|^{k-1})$ as $|t| \rightarrow 0$ through a set of Lebesgue density 1 at x_0 , we say f has an approximate $k-1$ derivative at x_0 . Finally, if $R_{x_0}(t) = o(|t|^{k-1})$ for each x_0 in a set E provided $t \rightarrow 0$ through a set whose complement has measure zero, we say f has a $(k-1)_*$ derivative for each x_0 in E .

With these remarks we can state the following known results.

THEOREM A (SEE [6]). *f has a k th derivative in the L^2 sense for almost every x_0 in E , $|E| > 0$, if and only if for almost every x_0 in E there exists an $\eta_{x_0} = \eta > 0$ such that*

$$(2) \quad \int_{|t| \leq \eta} \frac{|\Delta_{x_0}^k(t)|^2}{|t|^{n+2k}} dt < \infty.$$

THEOREM B. *f has a k th derivative in the L^p sense ($2 \leq p < \infty$) at almost every x_0 in E , $|E| > 0$, if and only if for almost every x_0 in E there exists $\eta_{x_0} = \eta > 0$ such that*

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$$(3) \quad \int_{|t| \leq \eta} \frac{|\Delta_{x_0}^k(t)|^2}{|t|^{n+2k}} dt < \infty, \quad \int_{|t| \leq \eta} \frac{|\Delta_{x_0}^k(t)|^p}{|t|^{n+pk}} dt < \infty.$$

Theorem A is due to Stein and Zygmund and was originally proved in [5] for the case of a function of a single variable. In [6] it is shown how it can be extended as indicated here. Theorem B was proved in [7] for the case $k=1$. With Theorem A one can prove Theorem B by using a similar argument to that found in [5, p. 280].

The theorem we wish to prove is the following one.

THEOREM 1. *f is equivalent to a function which has a k th derivative almost everywhere in a set E , $|E| > 0$, if and only if for almost every x in E there is an $\eta_x = \eta > 0$ such that the sequence*

$$(4) \quad \left\{ \int_{|t| \leq \eta} \frac{|\Delta_{x_0}^k(t)|^p}{|t|^{n+pk}} dt \right\}^{1/p}, \quad p = 2, 3, \dots,$$

is bounded.

2. We need some definitions and lemmas to enable us to prove these theorems. Let $E_y = \{x: f(x) > y\}$ for real values of y . Set $g(x_0) = \inf_y \{y: |E_y \cap Q| = o(|Q|) \text{ as } |Q| \rightarrow 0, x_0 \in Q\}$. Then $g: E_n \rightarrow R$ is a function, is called the upper boundary of f (see [1]), and satisfies the properties:

(P₁) $f(x) = g(x)$ at every point of approximate continuity of f , and hence f is equivalent to g ;

(P₂) for x_0 in E_n and $\epsilon > 0$, $\eta > 0$, the set $\{x: |x - x_0| < \eta \text{ and } |g(x) - g(x_0)| < \epsilon\}$ has positive measure.

LEMMA 1. *Let E be a measurable set and suppose f has a $(k)_*$ derivative at each point of E . Then f is equivalent to a function which has a k th derivative at each point of E .*

PROOF. Let g be the upper boundary of f . Then f is equivalent to g and $f(x) = g(x)$ at every point of E . Let x_0 be in E and let $\epsilon > 0$ be given. We may assume $P_{x_0}(t) = 0$. There is a $\delta > 0$ and a set N , $|N| = 0$, and $N \subset \{x: |x - x_0| < \delta\}$ such that $|g(x_0 + t)| \leq \epsilon |t|^k$ provided $x_0 + t \notin N$ and $|t| < \delta$. If $N \neq \emptyset$, there is a $\sigma > 0$ and a point t_0 , $|t_0| < \delta$, and $(x_0 + t_0) \in N$ such that $|g(x_0 + t_0)| \geq (\epsilon + \sigma) |t_0|^k$. By property (P₂) there is a sequence $\{t_n\}$, $t_n \rightarrow t_0$ and $x_0 + t_n$ is not in N , such that $g(x_0 + t_n) \rightarrow g(x_0 + t_0)$. This implies $|g(x_0 + t_0)| \leq \epsilon |t_0|^k$ from which we conclude $N = \emptyset$, and the lemma follows.

LEMMA 2. *Assume that for $x \in E$, $|E| > 0$, there is a set N_x , $|N_x| = 0$, such that $\Delta_x^k(t) = O(|t|^k)$ as $|t| \rightarrow 0$ for t in N_x . Then there exists a set N ,*

$|N| = 0$, such that $\Delta_x^k(t) = O(|t|^k)$ as $|t| \rightarrow 0$ with $(x \mp t) \notin N$ for each x in E .

PROOF. Let N be the set of points at which f is not approximately continuous. $|N| = 0$, and it can easily be shown that $\Delta_x^k(t) = O(|t|^k)$ as $|t| \rightarrow 0$ with $(x \mp t) \notin N$ for each x in E .

LEMMA 3. Let $f: E_n \rightarrow R$ be measurable and $1 \leq p \leq \infty$. Then at all points x , at which f has a k th derivative in the L^p sense, f has an approximate k th derivative.

LEMMA 4. Let $f: E_n \rightarrow R$ be measurable such that at each x_0 in a set E , $|E| > 0$, f has a k th approximate derivative and that the polynomial $P_{x_0}(t)$ is zero. Suppose that for each x_0 in E there is a set N_{x_0} such that $|N_{x_0}| = 0$ and $\Delta_{x_0}^k(t) = O(|t|^k)$ as $|t| \rightarrow 0$, $t \notin N_{x_0}$. Then f has a $(k)_*$ derivative at almost all $x_0 \in E$.

PROOF. As in Lemma 2, there is a set N , $|N| = 0$, such that $\Delta_x^k(t) = O(|t|^k)$ as $|t| \rightarrow 0$ with $x \pm t \notin N$ for each $x \in E$. Let $F_1 \subset F_2 \subset \dots$ be a sequence of closed sets such that f restricted to F_i is continuous, $F_i \subset E_n - N$, and $H = \bigcup_{i=1}^\infty F_i$ has full measure in every cube. Let

$$E(m, j) = \{x \in E \cap H: |\Delta_x^k(t)| < m|t|^k, 0 < |t| < 1/j, x + t \in H\}.$$

One can show these sets are measurable (see e.g. [4]). We will show that at each point of density of $E(m, j)$ f has a $(k)_*$ derivative, and this will complete the proof.

Assume x_0 is a point of density of $E(m, j)$. Let $\sigma > 0$ be given. Then there is a set $E_1 \subset E(m, j)$ with x_0 as a point of density, and if $x_0 + t$ is in E_1 we have $|f(x_0 + t)| \leq \sigma|t|^k$. Let

$$A_t = \left\{x: (x_0 + x + t)/2 \in E_1, 0 < |x - x_0| < |t| \right. \\ \left. \text{and } |x_0 + t - x| < (\sigma/m)^{1/k}|t| \right\}.$$

Let χ be the characteristic function of E_1 , then A_t has measure equal to

$$\int_{|x_0+t-x| < (\sigma/m)^{1/k}|t|} \chi\left(\frac{x_0 + x + t}{2}\right) dx.$$

Since x_0 is a point of density of the set E_1 , this integral is asymptotically equal to $C_n(\sigma/m)^{n/k}|t|^n$ as $|t| \rightarrow 0$ where C_n is a constant depending only on n . Hence there is $0 < \delta < 1/j$ such that $A_t \cap E \neq \emptyset$ whenever $0 < |t| < \delta$. Thus for every $|t|$, $0 < |t| < \delta$, there is an $x \in A_t \cap E_1$. Since $x \in E_1$ we have $|f(x)| \leq \sigma|x - x_0|^k \leq \sigma|t|^k$ and if $x_0 + t$ is in H we obtain

$$\begin{aligned} |f(x_0 + t)| &\leq |f(x_0 + t) + (-1)^{k-1}f(x)| + |f(x)| \\ &\leq m|(x_0 + t - x)/2|^k + \sigma|t|^k \\ &\leq m(\sigma/m)|t|^k + \sigma|t|^k = 2\sigma|t|^k. \end{aligned}$$

Hence f has a k th derivative at x_0 relative to the set H .

LEMMA 5 (SEE [2, p. 18]). *Let f be a real valued function in $L^p(E_n)$ ($1 \leq p \leq \infty$) and suppose f has a k th derivative in the L^p sense at each point of a set E of positive measure. Then, for $\epsilon > 0$, there is a closed set S , $|E - S| < \epsilon$, and two functions f_1 and f_2 such that $f = f_1 + f_2$ with $f_1 \in C^k$, and for x_0 in S ,*

$$\int_{|t| \leq \rho} |f_2(x_0 + t)|^p dt = o(\rho^{p(k+n)}), \quad \rho \rightarrow 0.$$

3. We are now ready to give the proof of Theorem 1. First let us assume that there is a function g equivalent to f which has a k th derivative almost everywhere in E . By Theorem A, for almost every x in E , there is an $\eta_x = \eta > 0$ such that

$$(5) \quad \int_{|t| \leq \eta} \frac{|\epsilon_x^k(t)|^2}{|t|^n} dt < \infty.$$

We will show that the sequence (4) is bounded at every point x for which $f(x) = g(x)$, g has a k th derivative and (5) holds. Since g has a k th derivative, $\Delta_x^k(g, t) = o(|t|^k)$. Also, since $f(x) = g(x)$ almost everywhere, there is an η' , $0 < \eta' < \eta$, and a set N_x of zero measure such that $\Delta_x^k(t) < |t|^k$ for $|t| < \eta'$ and $t \notin N_x$. Thus for $p \geq 2$

$$\int_{|t| \leq \eta'} \frac{|\epsilon_x^k(t)|^p}{|t|^n} dt \leq 2^{2-p} \int_{|t| \leq \eta'} \frac{|\epsilon_x^k(t)|^2}{|t|^n} dt.$$

Hence, (4) is bounded with η replaced by η' .

Now assume (4) is bounded, with $M < \infty$ as bound. Then we have $\text{ess sup} |\Delta_{x_0}^k(t)| \leq M|t|^k$, $0 < |t| < \eta$. The finiteness of the first term of the sequence ($p=2$) implies the existence of the k th derivative in the L^2 sense almost everywhere in E and that f is in L^2 of a neighborhood of almost every point of E . Let $\epsilon > 0$ be given. By Lemma 5, there is a set S such that $f = f_1 + f_2$, $|E - S| < \epsilon$, f_1 is C^k , and for x in S ,

$$(6) \quad \int_{|t| \leq \rho} |f_2(x + t)|^2 dt = o(\rho^{2(k+n)}).$$

Since ϵ is arbitrary, it is enough to show that f_2 is equivalent to a

function which has k derivatives at almost every point of S . Equation (6) and Lemma 3 imply that f_2 has a k th approximate derivative at the points x_0 in S and that the polynomial $P_{x_0}(t)$ for f_2 is zero. Since f_1 is C^k , it follows that $\Delta_x^k(f_1, t) = o(|t|^k)$ and hence $\Delta_x^k(f_2, t) = O(|t|^k)$ provided $|t| < \eta$ and $t \notin N_x$, $|N_x| = 0$. Now apply Lemma 4 and Lemma 1.

We point out that the boundedness of (4) and Theorem B imply that f has a k th derivative in the L^p sense ($2 \leq p < \infty$) almost everywhere in E . The boundedness of (4) may be regarded as a condition for the existence of the k th derivative in the L^p sense ($2 \leq p < \infty$) almost everywhere in E and uniformly in p . This is precisely what is needed to pass to the $p = \infty$ case, as was pointed out in [3] for a function of a single variable.

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DE PAUL UNIVERSITY