TYCHONOFF CUBES ARE COSET SPACES

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1. **Introduction.** Let T denote a Tychonoff cube (some infinite product of unit intervals) and let Q denote the Hilbert cube (the countably infinite product of intervals). It has been known for some time that Q is a homogeneous space. Consequently T is also homogeneous, since it can be written as a product of Hilbert cubes. In this note we shall show that T is a coset space.

2. **Preliminaries.** The question of when a homogeneous space X is a coset space has been considered by Ford [1] and Mostert [4], among others. Ford showed that it is necessary that X be completely regular and Hausdorff, and he proposed a very interesting sufficient (but not necessary) condition. Call an open set U in X an SLH set provided that for each $x, y \in U$ there is a $g \in G(X)$, the group of homeomorphisms of X, such that $g(x) = y$ and $g(z) = z$ for all $z \in X \setminus U$. The space X is called a strongly locally homogeneous (SLH) space if the SLH sets form a basis for the topology on X. Mostert stated the correct theorem using the SLH property.

THEOREM 1. *Let X be a completely regular Hausdorff space which is homogeneous and SLH. Then the topology on $G(X)$ induced by the uniformity on X obtained from the Stone-Čech compactification of X is reasonable in the sense that $G(X)$ is a topological group and the natural map $\mu: G(X)/G(X)_x \to X$ is a homeomorphism, where $G(X)_x$ denotes the stable group at some $x \in X$.*

We shall make use of the following observation.

LEMMA 2. *If X is homogeneous and the SLH sets in X contain a basis for the topology on X at some point $x \in X$, then X is an SLH space.*

PROOF. Let U be an open set containing $y \in X$. Let $h \in G(X)$ such that $h(y) = x$. Let V be an SLH set containing x such that $V \subset h(U)$. Clearly $h^{-1}(V)$ is an SLH set containing y and lying in U.

3. **T is an SLH space.** A property of $G(Q)$ discovered by R. Wong [5] provides the key to our argument. A mapping $H: I \times Q \to Q$ is called an isotopy if $H_t = H | t \times Q$ is in $G(Q)$ for all $t \in I$. When H is an isotopy, then H_0 and H_1 are said to be isotopic.

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THEOREM 3 (WONG). *Every element of $G(Q)$ is isotopic to the identity of $G(Q)$.*

THEOREM 4. *T is an SLH space.*

PROOF. Let $T=\prod_{\beta\in\Lambda} I_\beta$ and let $\mathcal{B}$ consist of the collection of open sets of the form

$$\prod_{\beta\in F}\left(\tfrac{1}{2}-\tfrac{1}{2}^n,\tfrac{1}{2}+\tfrac{1}{2}^n\right)\times\prod_{\beta\in\Lambda\setminus F} I_\beta,$$

where F is a finite subset of Λ and n is a natural number. Then $\mathcal{B}$ forms a basis for T at the point m, each of whose coordinates is $\frac{1}{2}$. We will show that each member of $\mathcal{B}$ is an SLH set. The theorem then follows from Lemma 2.

Choose

$$B=\prod_{\beta\in F} I_{\beta n}\times\prod_{\beta\in\Lambda\setminus F} I_\beta\in\mathcal{B},$$

where

$$F=\{1,2,\cdots,r\}\subset\Lambda\quad\text{and}\quad I_{\beta n}=\left(\tfrac{1}{2}-\tfrac{1}{2}^n,\tfrac{1}{2}+\tfrac{1}{2}^n\right)\subset I_\beta.$$

Then we may write

$$T=I^{r-1}\times I_r\times\prod_{\beta\in\Omega} Q_\beta$$

and

$$B=U^{r-1}\times I_{rn}\times\prod_{\beta\in\Omega} Q_\beta,$$

where

$$I^{r-1}=\prod_{\beta\in F\setminus r} I_\beta,\qquad U^{r-1}=\prod_{\beta\in F\setminus r} I_{\beta n},\quad\text{and}\quad\prod_{\beta\in\Omega} Q_\beta=\prod_{\beta\in\Lambda\setminus F} I_\beta.$$

Choose $x\in B$. We wish to construct a homeomorphism $h\in G(T)$ such that $h(x)=m$ and $h(z)=z$ for all $z\in T\setminus B$. This is done in several steps. For $A\subset\Omega$,

$$p_A\colon\prod_{\beta\in\Omega} Q_\beta\to\prod_{\beta\in A} Q_\beta$$

denotes the natural projection.

1. Let $f\in G(I^{r-1})$ such that

$$fp_{F\setminus r}(x) = p_{F\setminus r}(m) \quad \text{and} \quad f(z) = z$$

for all $z \in I^{r-1} \setminus U^{r-1}$. This is possible since U^{r-1} is an open $r-1$ cell contained in the interior of I^{r-1} and $p_{F\setminus r}(x)$, $p_{F\setminus r}(m)$ are in U^{r-1}.

2. Let $g \in G(I_r)$ such that $gp_r(x) = gp_r(m)$ and $g(z) = z$ for all $z \in I_r \setminus I_{rn}$.

3. For each $\beta \in \Omega$, let $g_\beta \in G(Q_\beta)$ such that $g_\beta p_\beta(x) = p_\beta(m)$. The homogeneity of each Q_β assures the existence of g_β.

4. For each $\beta \in \Omega$, use Theorem 3 to construct an isotopy $H^\beta: I_r \times Q_\beta \to Q_\beta$ so that $H_t^\beta = \mathrm{id} Q_\beta$, the identity on Q_β, for $t \in I_r \setminus I_{rn}$ and $H_{1/2}^\beta = g_\beta$.

5. Define

$$k: I_r \times \prod_{\beta \in \Omega} Q_\beta = I_r \times \prod_{\beta \in \Omega} Q_\beta$$

by

$$k(t \times y) = t \times \prod_{\beta \in \Omega} H_t^\beta p_\beta(y).$$

Note that $k \in G(I_r \times \prod_{\beta \in \Omega} Q_\beta)$ and $k(z) = z$ for

$$z \in \left(I_r \times \prod_{\beta \in \Omega} Q_\beta\right) \Big\backslash \left(I_{rn} \times \prod_{\beta \in \Omega} Q_\beta\right).$$

6. Now let $h \in G(T)$ be defined by $h = (f \times k) \circ g^*$ where g^* denotes the element of $G(T)$ given by $p_\beta g^*(z) = p_\beta(z)$ for $\beta \neq r$ and $p_r g^*(z) = gp_r(z)$. One may check that h is the required homeomorphism.

Since every point of B can be moved to m by a homeomorphism leaving $T \setminus B$ point wise fixed, it follows that B is an SLH set. This completes the proof.

The following is an immediate corollary of Theorem 1 and Theorem 4.

Corollary 5. *T is a coset space.*

Remark 6. It is interesting that although T is a coset space, it is not a coset space of any compact group. This follows from a theorem of Borel which states that an acyclic coset space of a compact group is a point [**2**, p. 310].

A theorem due to Madison [**3**] states that any continuous associative multiplication with identity on a continuum which admits a group structure is itself a group multiplication. It can easily be seen that T admits a continuous associative multiplication with identity which is not a group multiplication. Hence we have an example to show that "group structure" can not be weakened to "coset space" in Madison's theorem.

References

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