# MODULES OVER AN INCOMPLETE DISCRETE VALUATION RING[1]

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Throughout this note $R$ will denote a discrete valuation ring with unique prime ideal $(p)=Rp$. $Q$ will denote the quotient field of $R$ and $R^*$ will denote the completion of $R$. For basic definitions related to $R$-modules see [1], [2], or [5]. By rank, we shall mean torsion-free rank. If $M$ is an $R$-module and $x\in M$, $U_M(x)$ will denote the Ulm sequence $(\alpha_0, \alpha_1, \cdots, \alpha_n, \cdots)$ where, for each $n$, $\alpha_n$ is the height of $p^n x$. Two such sequences $(\alpha_0, \alpha_1, \cdots)$ and $(\beta_0, \beta_1, \cdots)$ are said to be equivalent if there is an $m$ and an $n$ such that $\alpha_{n+i}=\beta_{m+i}$ for all nonnegative integers $i$. If $M$ has rank one, then $U_M(x)$ and $U_M(y)$ are equivalent whenever $x$ and $y$ are elements of $M$ having zero order ideal. Consequently, if $M$ has rank one, we associate with $M$ an equivalence class $U(M)$ of Ulm sequences. $M_t$ will denote the torsion submodule of $M$. This note is devoted to establishing the following

THEOREM. *If $M$ and $N$ are countably generated rank one $R$-modules, then $M\cong N$ if and only if $M_t\cong N_t$ and $U(M)=U(N)$.*

This theorem has been proved by Kaplansky and Mackey [2] for the case when $R$ is complete and by Rotman [5] for the case when $p^\omega M=\bigcap_{n<\omega}p^n M=0$. Our extremely simple proof relies on the version already established by Kaplansky and Mackey and depends heavily on the imbedding of a module in its cotorsion completion. The basic facts that we require are the following: If $T=M_t$ and $M/T\cong Q$, then $M$ can be imbedded as a submodule of $T^*=\operatorname{Ext}_R^1(Q/R, T)$ such that $T^*/M$ is torsion-free and divisible. Moreover, $T^*$ is an $R^*$-module in a canonical fashion with $T$ the torsion $R^*$-submodule of $T^*$ (see [3]).

In proving our theorem, there is, of course, no loss in generality in assuming $M_t$ to be reduced. As it is clearly the case that $(\infty, \infty, \cdots, \infty, \cdots)\in U(M)$ corresponds to $M\cong M_t\oplus Q$, we may assume that $M$ is reduced. Now a torsion-free rank one $R$-module is isomorphic either to $R$ or to $Q$. Thus $M/M_t\cong R$, $M\cong M_t\oplus R$, and $(0, 1, 2, 3, \cdots, n, \cdots)\in U(M)$ are easily seen to be equivalent. Therefore

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we have reduced the proof to the case where $M$ is reduced and $M/M_t \cong Q$.

Henceforth we assume that $M_t = T = N_t$, $M/T \cong Q \cong N/T$, and that $M$ and $N$ are submodules of $T^*$ with $T^*/M$ and $T^*/N$ torsion-free and divisible. Let $M^*$ be the $R^*$-submodule of $T^*$ generated by $M$. Then $M^*$ is a rank one $R^*$-module and $M^*/M$ is torsion-free as an $R$-module. $M^*/M$ being torsion-free implies that $U(M^*) = U(M)$.

Lemma. *If $K$ is an $R$-submodule of $M^*$ such that $K/T \cong Q$, then $K \cong M$.*

Proof. Let $x \in M$ have zero order ideal. Since $T \subset K \subset M^*$, there is a nonzero $\pi_1 \in R^*$ with $\pi_1 x \in K$. Write $\pi_1 = p^n \pi$, where $\pi$ is a unit in $R^*$. Now $K$ is pure in $M^*$, for $K/T \cong Q$ implies $0 \to K/T \to M^*/T \to M^*/K \to 0$ splits, whence $M^*/K$ is torsion-free. Therefore $p^n M^* \cap K = p^n K$, and $\pi_1 x = p^n k_1$ for some $k_1 \in K$; thus $\pi x - k_1 \in T \subset K$ and $\pi x \in K$. Since $\pi$ is a unit, multiplication by $\pi$ is an automorphism of $T^*$. We claim that $\pi | M$ is an isomorphism of $M$ onto $K$.

First we show that $\pi$ maps $M$ into $K$. Let $y \in M$ be an element having zero order ideal. Choose nonzero elements $r$, $s \in R$ such that $rx = sy$. Then there is a $k \in K$ such that $r(\pi x + T) = s(k + T)$ and $s(\pi y + T) = s(k + T)$. Since $T^*/T$ is torsion-free, $\pi y - k \in T$ and $\pi y \in K$. Next we show that $\pi$ maps $M$ onto $K$. Let $z \in K$. We may assume that $z$ has zero order ideal. If $r$ and $s$ are nonzero elements of $R$ such that $rz = s\pi x$, we choose $m \in M$ such that $s(x + T) = r(m + T)$. Then $r(\pi m + T) = r(z + T)$ and $\pi m - z = t \in T$. Since $\pi$ is an automorphism of $T$, there is a $t_1 \in T$ such that $\pi t_1 = t$ and $z = \pi(m - t_1)$. Finally $\pi$ is univalent on $M$ since it is univalent on $T$.

The proof that $M \cong N$ if $U(M) = U(N)$ is easily completed now. We let $N^*$ be the $R^*$-submodule of $T^*$ generated by $N$. Since $U(N^*) = U(N) = U(M) = U(M^*)$, the Kaplansky-Mackey version of our theorem yields an $R^*$-isomorphism $\phi$ of $N^*$ onto $M^*$. If $K = \phi(N)$, it is easily seen that $K$ is an $R$-submodule of $M^*$ such that $K/T \cong Q$. Thus $M \cong K = \phi(N) \cong N$.

We mention in conclusion that the countability hypothesis is essential to the theorem as an example in [4] shows and that an existence theorem for countably generated modules of rank one over a discrete valuation ring has been established in [6] (such an existence theorem can also be obtained using properties of the module $T^*$).

## References

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