GROUP ALGEBRAS WITH NILPOTENT UNIT GROUPS

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In this note we determine under what conditions the group of units in the group algebra of a finite group is nilpotent.

Let G be a finite group, let F be a field of characteristic p, and let $F(G)$ denote the group algebra of G over F. The group of units in $F(G)$ will be denoted by $U=U(G, F)$.

THEOREM. (a) *Let $F(G)$ be semisimple. Then U is nilpotent if and only if G is abelian.*

(b) *Let p be a prime dividing the order of G. Then U is nilpotent if and only if G is nilpotent with abelian q-Sylow subgroup for each prime $q \neq p$.*

PROOF. (a) Suppose that G is nonabelian. Since $F(G)$ is semisimple, we have

$$F(G) \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_s}(D_s),$$

where each D_i is a division algebra over F; $M_{n_i}(D_i)$ denotes the total matric algebra over D_i.

$U(G, F)$ is thus the direct product of the general linear groups $\mathrm{GL}(n_i, D_i)$, $i=1, \cdots, s$.

Suppose that $p \neq 0$. If $n_i=1$, then the division algebra D_i is spanned by a homomorphic image of G; hence by [2], D_i is a field. Thus some n_j exceeds 1. For such an n_j, $\mathrm{GL}(n_j, D_j)$ is not nilpotent; hence U is not nilpotent.

Suppose that $p=0$. If some n_i is greater than 1, then as above, U is not nilpotent. Assume that each $n_i=1$. Then $F(G)$ is the direct sum of division algebras, so according to [1], G is Hamiltonian and U has the group of nonzero rational quaternions as a subgroup. It is easy to see that this group is not nilpotent; hence neither is U.

(b) The following easy result gives a convenient way of handling the sufficiency part of (b) when F is infinite.

LEMMA. *Let R be a ring with identity. Suppose that R has a nilpotent ideal N such that R/N is commutative. Then the group V of units in R is nilpotent.*

PROOF. For x and y in R, we have by hypothesis that the Lie product $[x, y]=xy-yx$ is in N.

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For u and v in V, define $(u, v) = u^{-1}v^{-1}uv$; for elements $u_1, \cdots, u_n$ in V, define inductively $(u_1, \cdots, u_n) = ((u_1, \cdots, u_{n-1}), u_n)$. Using the equations

$$(u_1, \cdots, u_n) = 1 + (u_1, \cdots, u_{n-1})^{-1} u_n^{-1} [(u_1, \cdots, u_{n-1}), u_n]$$

and

$$[x - 1, y] = [x, y],$$

we see by induction that for $n > 1$ and for $u_1, \cdots, u_n$ in V, $(u_1, \cdots, u_n) - 1$ is in N^{n-1}. Since N is a nilpotent ideal, we have that V is a nilpotent group. This proves the lemma.

If U is nilpotent, then so is G. Assume this to be the case and let A be the p-complement of G. Then $U(A, F)$ is a subgroup of U and hence is nilpotent. By part (a), A is abelian.

Conversely, let $G = P \times A$, where P is a p-group (perhaps trivial) and A is abelian of order prime to p. Suppose that $F(A) = F_1 \oplus \cdots \oplus F_k$, where each F_i is a field over F. Then

$$F(G) \cong F_1 \otimes F(P) \oplus \cdots \oplus F_k \otimes F(P).$$

As a ring, each $F_i \otimes F(P) \cong F_i(P)$. Since P is a p-group, $F(G)$ modulo its radical is isomorphic to $F(A)$. Thus by the lemma, U is nilpotent.

References

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