SPECIAL DIVISORS ON COMPACT RIEMANN SURFACES

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Introduction. In a previous paper [1] the author obtained some results concerning the distribution of special divisors on compact Riemann surfaces of genus $g$. The reader is referred to [1] for definitions and notation.

It was shown in [1, p. 886] that the product structure with $S_r \times S_r$ and $T^v(S)$ as factors may be endowed with a structure of complex analytic manifold in such a way that the resulting space, $W_{r,r}$, is an analytic fibre space over the base manifold $T^v(S)$ with fibre $S_r \times S_r$ over $S(T) \subseteq T^v(S)$. We then proved [1, Theorem 2] that if $\xi$, $\omega$ are completely distinct equivalent special divisors of degree $r$ on $S_0$, then considering the triple $(\xi, \omega, S_0(T))$ as a point in $W_{r,r}$, there is a $g$ codimensional submanifold of $W_{r,r}$ containing the point $(\xi, \omega, S_0(T))$ (each point of which has projections onto pairs of equivalent, special divisors on the surface $S_0(T)$, the base point under the fibre) which projects onto a $\lambda$ codimensional submanifold of $T^v(S)$. Bounds were obtained for $\lambda$, and from these bounds it followed that a special divisor of degree $r < (g+1)/2$ is always special in the sense of moduli. Finally we showed that if $g$ is odd, a special divisor of degree $(g+1)/2$ is also special in the sense of moduli. Hence our results for special divisors were [1, Theorem 5] that if $g$ is even (odd), a special divisor of degree less than $(g+2)/2$ ($(g+3)/2$) is special in the sense of moduli. As a particular example of the method, we computed the dimension of the sublocus of $T^v(S)$ possessing Weierstrass points whose Weierstrass sequences begin with a fixed $r < g$.

It is the purpose of this note to indicate that the techniques used in [1] can be employed to yield a general theorem from which the results of [1] emerge as corollaries. Furthermore, in our present treatment, we shall obtain directly that a special divisor of degree less than $(g+2)/2$ is special in the sense of moduli, eliminating the necessity of Theorem 4 in [1].

1. Suppose $f$ is a meromorphic function with $r$ zeros and $r$ poles on $S$ a compact Riemann surface of genus $g$. Then, by Abel’s theorem, $S$ possesses a pair of integral, equivalent, completely distinct, special divisors of degree $r$, and $f$ projects $S$ onto an $r$-sheeted branched cover

Received by the editors January 13, 1967.

1 The research was partially supported by the NFS GP-3452.
of the Riemann sphere with $2r + 2g - 2$ branch points. Conversely, if $S$ permits a representation as an $r$-sheeted cover of the sphere, $S$ possesses a pair of equivalent, integral, completely distinct, special divisors of degree $r$. Hence we see that the existence of a pair of integral, equivalent, completely distinct, special divisors on $S$ is equivalent to the existence of an $r$-sheeted concrete representation of the surface $S$. With this in mind, the results of [1] mentioned in the introduction can be restated in the following way: for $g$ even (odd) the property of permitting an $r$-sheeted representation is special in the sense of moduli if $r < (g+2)/2 ((g+3)/2)$.

Having stated the results in this form, the following question naturally presents itself. Let $S$ be a compact Riemann surface of genus $g$ which permits an $r$-sheeted concrete representation with an $(s-1)$th order branch point $2 \leq s \leq r$. When is such a representation special in the sense of moduli? Before we answer this question, we observe that the desired property is equivalent to the demand that $S$ possess a pair of integral, equivalent, special divisors $\zeta = P^s P_1 \cdots P_{r-s}, \omega = Q_1 \cdots Q_r$. This follows by virtue of the fact that we can always arrange that the branch point of order $s-1$ lie over the origin or $\infty$.

We recall that an integral divisor $\zeta$ of degree $r < g$ is said to be special if $i(\zeta) = g - r + 1 + m, m \geq 0$.

**Theorem 1.** The dimension $d$ of the locus of $T_0(S)$ whose underlying surfaces permit $r$-sheeted representations with an $(s-1)$th order branch point satisfies the inequalities

$$r + 2g - 4 - m \leq d \leq \min(V - m - (s + 1), 3g - 3)$$

where $V = 2r + 2g - 2$ and $m$ is computed from the index of specialty of the divisor.

**Proof.** As we have already indicated, the desired property is equivalent to the existence on $S$ of integral, equivalent, completely distinct, special divisors $\zeta = P^s P_1 \cdots P_{r-s}, \omega = Q_1 \cdots Q_r$.

By Abel’s theorem

$$u_i(\zeta) - u_i(\omega) - m_i - \sum_{j=1}^{g} n_j \Pi_{ij} = 0, \quad i = 1, \cdots g,$$

where $m_i, n_j$ are integers. These equations are of course the same as

$$su_i(P) + u_i(P_1) + \cdots + u_i(P_{r-s}) - u_i(Q_1) - \cdots - u_i(Q_r) - m_i$$

$$- \sum_{j=1}^{g} n_j \Pi_{ij} = 0.$$
We now view these equations as ranging over $W_{r-s+1,r}$, and just as in the proof of Theorem 2 in [1], define therein a $g$ codimensional submanifold, each point of which has projections onto pairs of equivalent, integral, special divisors of the specified form which projects onto a $\lambda$ codimensional submanifold of $T^g(S)$. The same sort of estimates which yielded Theorem 3 in [1] give here that

$$g - 2r + m + s \leq \lambda \leq g - r + 1 + m$$

and we therefore immediately obtain the desired inequalities on $d$.

We now restrict ourself to the case $m=0$. If $m>0$, then $S$ permits a concrete representation with fewer than $r$ sheets. If $m=0$ the inequalities on $d$ are

$$r + 2g - 4 \leq d \leq \min(V - (s + 1), 3g - 3).$$

**Corollary** Let $S$ be a compact Riemann surface of genus $g$. The property of $S$ permitting an $r$-sheeted concrete representation with an $(s-1)$th order branch point is special in the sense of moduli if $r< (g+s)/2$.

**Proof.** By definition, the property is special if $d<3g-3$. $d<3g-3$ if $r< (g+s)/2$.

2. In [1] we considered arbitrary, integral, special divisors of degree $r<g$. These special divisors of course give rise to meromorphic functions which project the Riemann surface onto a concrete $r$-sheeted cover of the sphere with $2r+2g-2$ branch points. The point is, and this is what we omitted mentioning in [1], that even if the branch points are all simple we can still assume that the special divisor is of the form $P^2P_1 \cdots P_{r-2}$. Hence, we can apply our theorem with $s=2$ and obtain the result that a special divisor of degree $<(g+2)/2$ is always special in the sense of moduli, and we do not have to appeal to Theorem 4 of [1].

In the case of a Weierstrass point whose Weierstrass sequence begins with a fixed $r<g$, we have $s=r$ and hence the property is special if $r< (g+r)/2$ or if $r<g$. Furthermore, in this case, the inequalities satisfied by $d$ are

$$r + 2g - 4 \leq d \leq r + 2g - 3.$$ 

Returning for a moment to the case $s=2$ and $m=0$, we see that in this case the inequalities on $d$ are

$$r + 2g - 4 \leq d \leq \min(V - 3, 3g - 3).$$

Hence, we see that if there are few enough branch points, we obtain the result of Riemann that after normalizing 3 branch points the
number that remain are an upper bound for the dimension of the space of moduli. In connection with these remarks the reader is referred to [2] for a more complete discussion.

The author is indebted to Professor H. E. Rauch for many interesting and stimulating discussions concerning the research reported on in this note.

**References**


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