PR-FACTORIZATIONS OF FAMILIES OF LIGHT INTERIOR FUNCTIONS

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Introduction. A familiar technique used in dealing with problems which fall into a certain class is to find those problems which can be converted into some canonical form which is more easily analyzed. In this paper I shall be considering families of light interior functions defined in a domain $\mathcal{D}$ in $E^2$ and the canonical forms will be families of Bers functions.

1. Preliminary concepts and definitions. If $f$ is a $C^r$ function, the Jacobian matrix of $f$ will be denoted by $J(f)$ and the determinant of $J(f)$ will be denoted by $|J(f)|$.

Definition 1.1. A $C^r$ function $f$ will be said to be pseudo-regular in $\mathcal{D}$ if (i) $|J(f)| \geq 0$, (ii) $|J(f)| = 0$ if and only if $J(f)$ is the zero matrix, and (iii) the critical points of $f$ are countable and have no limit point in $\mathcal{D}$. A pseudo-regular function is locally quasiconformal except in a neighborhood of a critical point.

Definition 1.2. A collection $\mathcal{W}$ of light interior functions defined in $\mathcal{D}$ will be called a real linear family if for $f_1$ and $f_2$ in $\mathcal{W}$, $c_1f_1 + c_2f_2$ is in $\mathcal{W}$ for all real $c_1$ and $c_2$. $\mathcal{W}$ will be said to be nontrivial if it contains at least two linearly independent elements.

Definition 1.3. A real linear family $\mathcal{W}$ will be called a Bers family if the elements of $\mathcal{W}$ are solutions of a Bers system $U_x = \sigma V_x + \tau V_y$, $-U_y = \sigma V_x - \tau V_y$, where $\sigma$ and $\tau$ are Hölder-continuous real-valued functions and $\sigma > 0$. Elements of a Bers family will be called Bers functions. (Bers [1] calls these functions "pseudoanalytic functions of the second kind").

Definition 1.4. A real linear family $\mathcal{W}$ of light interior functions defined in $\mathcal{D}$ will be said to have a PR-factorization if there exists a homeomorphism $h$ defined in $\mathcal{D}$ and a Bers family $\mathcal{W}$ in $h(\mathcal{D})$ such that if $f$ is in $\mathcal{W}$, there exists $\tilde{f}$ in $\mathcal{W}$ for which $f = f \circ h$. We denote this PR-factorization by $[h, \mathcal{W}]$.

In an earlier paper [3], I showed that every nontrivial real linear family of pseudo-regular functions has a PR-factorization with $h$ a Beltrami function. Furthermore, this factorization is not unique and if $h_1$ and $h_2$ are Beltrami functions satisfying the same Beltrami system

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in $\mathcal{D}$ and corresponding to two distinct PR-factorizations of $\mathcal{W}$, $h_1 \circ h_2^{-1}$ is a conformal mapping of $h_2(\mathcal{D})$ onto $h_1(\mathcal{D})$.

2. Some theorems on PR-factorization. If $\mathcal{W}$ is a Bers family, we may assume that $\mathcal{W}$ contains at least one homeomorphism since every Bers system has homeomorphic solutions. It follows that if $\mathcal{W}$ is a maximal nontrivial real linear family which has a PR-factorization, $\mathcal{W}$ contains at least one homeomorphism. Finally, if $\mathcal{W}$ is a Bers family, $\mathcal{W}$ always has a PR-factorization of the type $[h, \tilde{\mathcal{W}}]$ where $h$ is conformal.

**Theorem 2.1.** Let $\mathcal{W}$ be a nontrivial real linear family. If $\mathcal{W}$ has a PR-factorization, this PR-factorization is not unique.

**Proof.** Let $[h, \tilde{\mathcal{W}}]$ be a PR-factorization of $\mathcal{W}$. Since $\tilde{\mathcal{W}}$ is a Bers family, $\tilde{\mathcal{W}}$ has a PR-factorization $[g, \tilde{\mathcal{W}}]$ where $g$ is conformal. If we define $h' = g \circ h$, $[h', \tilde{\mathcal{W}}]$ is a PR-factorization of $\mathcal{W}$.

If, in the preceding theorem, we let $h_1 = h$ and $h_2 = h \circ g$, we see that $h_1 \circ h_2^{-1} = g^{-1}$ and $h_2 \circ h_2^{-1} = g$ are conformal. In the general case, $h_1 \circ h_2^{-1}$ will not necessarily be conformal.

**Theorem 2.2.** Let $[h_1, \mathcal{W}_1]$ and $[h_2, \mathcal{W}_2]$ be PR-factorizations of a nontrivial real linear family $\mathcal{W}$ and let $\tau_i$ and $\sigma_i$, $i = 1, 2$, be the coefficients of the associated Bers systems. Then $h_1 \circ h_2^{-1}$ is pseudo-regular and if $h_1 \circ h_2^{-1}$ is conformal, $\tau_2 = \tau_1 \circ (h_1 \circ h_2^{-1})$ and $\sigma_2 = \sigma_1 \circ (h_1 \circ h_2^{-1})$.

**Proof.** Let $f_1$ be a homeomorphic element of $\mathcal{W}_1$ and let $f_2$ be the element of $\mathcal{W}_2$ such that $f_2 = f_1 \circ (h_1 \circ h_2^{-1})$. $f_1$ and $f_2$ are pseudo-regular on their respective domains. Since the composition of pseudo-regular functions is a pseudo-regular function and the inverse of a homeomorphic pseudo-regular function is pseudo-regular, $h_1 \circ h_2^{-1} = f_1^{-1} \circ f_2$ is pseudo-regular. Now let $h_1 \circ h_2^{-1} = p + iq$, let $f_1 = u + iv$, and let $f_2 = r + is$. A simple computation shows that

(2.1) $\tau_1(x, y) = \tau_2(p(x, y), q(x, y)) + A \sigma_2(p(x, y), q(x, y))$

and

(2.2) $\sigma_1(x, y) = B \sigma_2(p(x, y), q(x, y))$

where

(2.3) $A = \frac{s_p s_q (p_x^2 + p_y^2 - q_x^2 - q_y^2) - (s_p^2 - s_q^2)(p_x q_x + p_y q_y)}{s_p^2 (p_x^2 + p_y^2) + 2 s_p s_q (p_x q_x + p_y q_y) + s_q^2 (q_x^2 + q_y^2)}$

and
If \( h_1 \circ h_2^{-1} \) is conformal, \( A = 0 \) and \( B = 1 \).

Since pseudo-regular functions have many properties in common with analytic functions, one might suppose that the theorem on removable singularities could be extended to pseudo-regular functions. This is not true. One can exhibit functions which are homeomorphisms of \( \mathbb{E}^2 \) into \( \mathbb{E}^2 \), have partial derivatives at each point in \( \mathbb{E}^2 \), and are pseudo-regular in the punctured plane but which are not pseudo-regular at the origin. It is apparent from the preceding theorem and from earlier discussions that if \( \mathcal{W} \) has a PR-factorization and if \( \mathcal{W} \) contains a homeomorphic pseudo-regular element, then every element of \( \mathcal{W} \) is pseudo-regular and \( h \) may be taken to be a Beltrami function.

Theorem 2.3. Let \( \mathcal{W} \) be a nontrivial real linear family of light interior functions defined in \( \mathbb{D} \). If \( \mathcal{W} \) has a PR-factorization, either there exists a subset \( E \) of \( \mathbb{D} \) having no limit point in \( \mathbb{D} \) such that every element of \( \mathcal{W} \) is pseudo-regular in \( \mathbb{D} - E \) or \( \mathcal{W} \) contains no pseudo-regular elements.

Proof. Let \([h, \mathcal{W}]\) be a PR-factorization of \( \mathcal{W} \). If \( f \) is a pseudo-regular element of \( \mathcal{W} \), \( h \) is also pseudo-regular except possibly at the critical points of \( f \). Since every element of \( \mathcal{W} \) may be represented as the composition of \( h \) with a Bers function, every element of \( \mathcal{W} \) is pseudo-regular in \( \mathbb{D} - E \) where \( E \) is the set of critical points of \( f \). Finally, since \( f \) is pseudo-regular in \( \mathbb{D} \), \( E \) has no limit point in \( \mathbb{D} \).

Corollary 2.3. Let \( \mathcal{W} \) be a nontrivial real linear family of light interior functions defined in \( \mathbb{D} \). Let \( g \) be a pseudo-regular element of \( \mathcal{W} \), and let \( E \) be the critical points of \( g \). If \( \mathcal{W} \) contains a function which is not pseudo-regular in \( \mathbb{D} - E \), \( \mathcal{W} \) has no PR-factorization.

Proof. Trivial.

A family satisfying the hypotheses of this corollary will be exhibited in the next section.

3. A nontrivial family which has no PR-factorization. If \( f(x, y) = -y + ix \) and \( g(x, y) = x^3 + iy^3 \), \( f \) and \( g \) are \( C' \) homeomorphisms (\( f \) is analytic) and it is easy to verify that \( \alpha f + \beta g \) is a \( C' \) light interior mapping for \( \alpha \) and \( \beta \) arbitrary real numbers. Restricting ourselves for the moment to the open first quadrant, \( f \) and \( g \) determine the first order system

\[
B = \frac{(s_p^2 + s_q^2)(p_u q_v - p_v q_u)}{s_p^2(p_x^2 + p_y^2) + 2s_p s_q(p_u q_x + p_v q_y) + s_q^2(q_x^2 + q_y^2)}.
\]
(3.1) \[ U_x = (x^2/y^2)V_y, \quad -U_y = V_x. \]

Obviously, \( f \) and \( g \) are solutions of (3.1). By quite elementary methods, one may obtain a five-parameter family of solutions of (3.1) given by

\[
U = x^{3/2}y^{1/2}\left[ C_1I_{1/4}(\alpha y^2) - C_2I_{-1/4}(\alpha y^2) \right]
\cdot \left[ C_3J_{-3/4}(\alpha x^2) - C_4J_{3/4}(\alpha x^2) \right],
\]

\[
V = x^{1/2}y^{3/2}\left[ C_1I_{3/4}(\alpha y^2) + C_2I_{1/4}(\alpha y^2) \right]
\cdot \left[ C_3J_{1/4}(\alpha x^2) + C_4J_{-1/4}(\alpha x^2) \right]
\]

where \( C_1, C_2, C_3, \) and \( C_4 \) are arbitrary real numbers, \( \alpha \) is any positive real number, \( J_p \) is the Bessel function of order \( p \), and \( I_p \) is the modified Bessel function of the first kind of order \( p \). Note that \( U, V, \) and their partial derivatives have at most removable discontinuities at \( x=0 \) and \( y=0 \) and are continuous for all other values of \( x \) and \( y \). Furthermore,

\[
U_xV_y - U_yV_x = V_x^2 + \frac{x^2}{y^2}V_y^2
\]

\[
= 4\alpha^2x^3y^3\left[ C_1I_{-3/4}(\alpha y^2) + C_2I_{3/4}(\alpha y^2) \right]^2
\cdot \left[ C_3J_{-3/4}(\alpha x^2) - C_4J_{3/4}(\alpha x^2) \right]^2
+ 4\alpha^2xy\left[ -C_1I_{1/4}(\alpha y^2) + C_2I_{-1/4}(\alpha y^2) \right]^2
\cdot \left[ C_3J_{1/4}(\alpha x^2) + C_4J_{-1/4}(\alpha x^2) \right]^2
\]

is nonnegative and bounded for all finite values of \( x \) and \( y \). If \( \mathcal{W} \) consists of \( f, g, \) and functions of the form \( u + iv \) where the pair \( (u, v) \) satisfy (3.1), it contains a pseudo-regular function and functions which are not pseudo-regular.

References


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