

A GENERALIZATION OF A THEOREM OF HANS LEWY

HUGO ROSSI¹

Let ρ be a real-valued C^∞ function defined in a neighborhood of the origin 0 in C^n , such that $\rho(0) = 0$, $d\rho(0) \neq 0$. Then, near zero, $M = \{z; \rho(z) = 0\}$ is a real submanifold of C^n of dimension $2n - 1$. If $\partial\bar{\partial}\rho(0) \neq 0$, then M has a holomorphic hull which contains an open set. We shall prove an L^2 version of this fact. Let $\bar{\partial}_b$ represent the tangential Cauchy-Riemann operator on M introduced by Kohn [1]. By L^2 on M , we mean the space of functions which are square integrable with respect to surface area.

THEOREM. *There is a neighborhood N of 0 such that if $D = \{z \in N; \rho(z) < 0\}$ and f is an L^2 function on $N \cap M$, the following are equivalent:*

- (i) *f is a weak solution of the equation $\bar{\partial}_b f = 0$,*
- (ii) *$\int_M f \bar{\partial} \alpha = 0$ for all $(n, n-2)$ forms α whose support intersects $N \cap M$ in a compact set,*
- (iii) *f is the boundary value of a function holomorphic in D ,*
- (iv) *f is locally L^2 approximable by functions holomorphic in a neighborhood of $N \cap M$.*

The only nontrivial part of this theorem is (ii) implies (iii) and (iv); this was proven in [2] by Hans Lewy for f a C^1 function ($n = 2$, but that does not matter). The proof here is an adaptation of his argument. We need to use the following verifiable lemmas.

LEMMA 1. *Let f be a square integrable function defined in a domain in C^n . f is holomorphic if and only if $\int f \bar{\partial} \alpha = 0$ for all compactly supported $(n, n-1)$ forms α .*

LEMMA 2. *Let X be a compact Hausdorff space, μ a finite Baire measure, $\Delta = \{z \in C; |z| < 1\}$, $\Gamma = \{z \in C; |z| = 1\}$. Let $f: X \rightarrow L^2(\Gamma)$ be square integrable; $\int \|f(x)\|^2 d\mu < \infty$ and suppose also*

$$\int_{\Gamma} f(x)(\theta) e^{in\theta} d\theta = 0 \quad \text{for } n > 0.$$

Let $\hat{f}(x, z)$ for $z \in \Delta$ be the Cauchy integral of $f(x)$. Then \hat{f} has the bound-

Received by the editors November 30, 1966.

¹ This research was done while the author was a fellow of the Alfred P. Sloan Foundation.

ary values f . More precisely, let $\phi: X \times \Gamma \times [0, 1] \rightarrow \bar{\Delta}$ be a continuous map with these properties:

- (i) $\phi(x, \theta, t) = \phi(x, t)e^{i\theta}$, $\phi(x, 0) = 1$,
- (ii) $\phi(X \times \Gamma \times (0, 1)) \subset \Delta$.

Then

$$\lim_{\delta \rightarrow 0} \int_{X \times \Gamma} |\hat{f}(x, \phi(x, \theta, \delta)) - f(x, \theta)|^2 d\theta d\mu = 0.$$

Now, we return to M . Because $\partial\bar{\partial}\rho(0) \neq 0$ and $d\rho(0) \neq 0$ we may choose complex coordinates $(z, w, w_1, \dots, w_{n-2})$ near zero so that M is given by

$$0 = \rho(z) = \operatorname{Re} w + z\bar{z} + Q(\zeta, \zeta) + O(2),$$

where ζ is the multivariable (w_1, \dots, w_{n-2}) , Q is a quadratic form, and $O(2)$ consists of terms of higher order at 0. Let $\pi: C^n \rightarrow C^{n-1}$, $\pi(z, w, \zeta) = (w, \zeta)$. It is shown in [3], that the mapping π has the following structure. There is a ball N , center at 0 such that $\pi(M \cap N)$ is the closure (in $\pi(N)$) of a domain D_0 . For $(w, \zeta) \in D_0$, $\Gamma_{(w, \zeta)} = \pi^{-1}(w, \zeta) \cap M$ is a simple closed curve in the z -plane bounding the domain $\Delta_{(w, \zeta)}$. As $(w, \zeta) \rightarrow \partial D_0$, $\Gamma_{(w, \zeta)} \rightarrow \text{point}$. Let $D = \{(z, w, \zeta); (w, \zeta) \in D_0, z \in \Delta_{(w, \zeta)}\}$. With the situation so given we prove

LEMMA 3. Let $f \in L^2$ on M with the property (ii) of the theorem. Then

$$\hat{f}(z, w, \zeta) = \frac{1}{2\pi i} \int_{\Gamma_{(w, \zeta)}} \frac{f(\eta, w, \zeta) d\eta}{\eta - z}$$

is holomorphic in D . If B is a closed ball contained in N , $f|_B$ is L^2 -approximable by translates of \hat{f} which are holomorphic in a neighborhood of $B \cap M$.

PROOF. First of all, \hat{f} is clearly locally L^2 in D . We use Lemma 1 to verify that \hat{f} is holomorphic. Let β be a compactly supported (in D), $(n, n-1)$ form. Let $dV' = dw \wedge dw_1 \wedge \dots \wedge dw_{n-2} \wedge d\bar{w} \wedge \dots \wedge d\bar{w}_{n-2}$.

(i) $\beta = h dz \wedge dV'$. Then

$$\begin{aligned} \int \hat{f} \bar{\partial} \beta &= \int_{D_0} \left\{ \int_{\Delta_{(w, \zeta)}} \left[\frac{1}{2\pi i} \int_{\Gamma_{(w, \zeta)}} \frac{\partial h}{\partial \bar{z}} \frac{f(\eta, w, \zeta) d\eta}{\eta - z} \right] d\bar{z} \wedge dz \right\} dV' \\ &= \frac{1}{2\pi i} \int_{D_0} \left\{ \int_{\Gamma_{(w, \zeta)}} f(\eta, w, \zeta) \left[\int_{\Delta_{(w, \zeta)}} \frac{1}{\eta - z} \frac{\partial h}{\partial \bar{z}} d\bar{z} \wedge dz \right] d\eta \right\} dV'. \end{aligned}$$

Since η is outside the support of h , by Lemma 1 for $n=1$, the innermost integral is always zero. Thus $\int \hat{f} \bar{\partial} \beta = 0$ in this case.

(ii) $\beta = d\bar{z} \wedge dz \wedge \gamma$, where γ is compactly supported $(n-1, n-2)$

with no $d\bar{z}$ or dz term. Then $\bar{\partial}\beta = d\bar{z} \wedge dz \wedge \bar{\partial}'\gamma$ where $\bar{\partial}'\gamma$ is taken as if γ were considered as an $(n-1, n-2)$ form in the (w, ζ) -space, with coefficients varying in z .

Thus

$$\begin{aligned} \int \hat{f}\bar{\partial}\beta &= \frac{\pm 1}{2\pi i} \int_{D_0} \left\{ \int_{\Delta(w,\zeta)} \left[\int_{\Gamma(w,\zeta)} \frac{f(\eta, w, \zeta) d\eta}{\eta - z} \wedge \bar{\partial}'\gamma \right] d\bar{z} \wedge dz \right\} \\ &= \frac{\pm 1}{2\pi i} \int_{D_0} \left\{ \int_{\Gamma(w,\zeta)} f(\eta, w, \zeta) \left[\int_{\Delta(w,\zeta)} \frac{\bar{\partial}'\gamma}{\eta - z} d\bar{z} \wedge dz \right] d\eta \right\}. \end{aligned}$$

Let

$$\alpha(\eta, w, \zeta) = \left(\int_{\Delta(w,\zeta)} \frac{\gamma}{\eta - z} d\bar{z} \wedge dz \right) \wedge d\eta.$$

α is a $C^\infty(n, n-2)$ form defined in a neighborhood of M whose support intersects M in a compact set (since γ vanishes in a neighborhood of M). Further, computing $\bar{\partial}\alpha$, we find $\int \hat{f}\bar{\partial}\beta = \int_M f\bar{\partial}\alpha = 0$ by hypothesis.

Now since any compactly supported $(n, n-1)$ form on D is a sum of forms of type (i) and (ii), we have $\int \hat{f}\bar{\partial}\beta = 0$ for all such forms, so by Lemma 1, \hat{f} is holomorphic.

Now, in order to apply Lemma 2, we must verify that, for fixed (w, ζ) , $\hat{f}(z, w, \zeta)$ has the boundary value f . That is

$$(*) \quad \int_{\Gamma(w,\zeta)} f(z, w, \zeta) z^n dz = 0 \quad \text{for } n \geq 0.$$

Let

$$\begin{aligned} F(w, \zeta) &= \int_{\Gamma(w,\zeta)} f(z, w, \zeta) z^n dz \quad w \in D_0, \\ &= 0 \quad w \notin D_0. \end{aligned}$$

We show that F is holomorphic in $\pi(N)$. First, if β is a $C^\infty(n-1, n-2)$ form, compactly supported in D_0 ,

$$\int F\bar{\partial}\beta = \int_{D_0} \int_{\Gamma(w,\zeta)} f(z, w) z^n dz \wedge \bar{\partial}\beta = \int_M f\bar{\partial}(z^n dz \wedge \beta) = 0.$$

Let β now be any $(n-1, n-2)$ form compactly supported in $\pi(N)$. Choose real C^∞ coordinates in $\pi(N)$, x_1, \dots, x_{2n-2} so that $bD_0 = \{x_1 = 0\}$, $D_0 = \{x_1 > 0\}$. (We may have to do this locally, but after applying a partition of unity to β this is the general case.) Reducing to the plane $x_2, \dots, x_{2n-2} = \text{constant}$, $\arg z_1 = \text{constant}$, we see that M intersects this plane in a curve $x_1 = A |z_1|^2 + \dots$, where A depends

differentiably on the other constants and is bounded away from zero. Thus the length of Γ_w is of the order of $2\pi\sqrt{x_1}$.

Now let $\rho(x_1)$ be a C^∞ function such that

$$\begin{aligned} \rho &\equiv 0 && \text{when } x_1 \geq \epsilon, \\ \rho &\equiv 1 && \text{when } x_1 \leq 0, \end{aligned} \quad \left| \frac{d\rho}{dx_1} \right| \leq 2/\epsilon.$$

Now $\int_{D_0} F\bar{\partial}\beta = \int_{D_0} F\bar{\partial}(\rho\beta)$, since $\int F\bar{\partial}(1-\rho)\beta = 0$, as above. Now $\bar{\partial}\rho\beta = \bar{\partial}\rho \wedge \beta + \rho\bar{\partial}\beta$,

$$\begin{aligned} \left| \int F\bar{\partial}\rho \wedge \beta \right| &= \left| \int_{D_0} \int_{\Gamma(w,\zeta)} f(z, w, \zeta) dz \wedge \bar{\partial}\rho \wedge \beta \right| \\ &= \left| \int_{D_0} \int_{\Gamma(w,\zeta)} \left\{ f(z, x) z^n \frac{d\rho}{dx_1} c_\beta(x) \right\} dz \wedge dV \right| \end{aligned}$$

where c_β is a function depending only on x , and dV is the element of volume in D_0 . Using Schwarz's inequality,

$$\left| \int F\bar{\partial}\rho \wedge \beta \right| \leq K_n \|f\| \left(\int_{D_0} \left| \frac{\partial\rho}{\partial x_1} c_\beta(x) \right|^2 \int_{\Gamma(w,\zeta)} d|z| \right)^{1/2}$$

But $\int_{\Gamma_w} d|z| \sim 2\pi\sqrt{x_1}$, thus

$$\left| \int F\bar{\partial}\rho \wedge \beta \right| \leq K_n' K_\beta \|f\| \epsilon^{-1} \int_{0 \leq x_1 \leq \epsilon} \sqrt{x_1} dx_1 \leq \|f\| \sqrt{\epsilon}.$$

Now $\int F\rho\bar{\partial}\beta$ is even better, so we find, letting $\epsilon \rightarrow 0$ that $\int_{D_0} F\bar{\partial}\beta = 0$. Thus F is holomorphic in $\pi(N)$, and since it is identically zero in an open set, it is identically zero, and (*) is verified.

Now, fix a Riemann map $R_{(w,\zeta)}$ of $\Delta_{(w,\zeta)}$ onto $\{|z| < 1\}$, differentiable at the boundary, and varying differentiably in (w, ζ) . Define $\psi: \bar{D} \rightarrow M$, $\psi(z, w, \zeta)$ = the point on $\Gamma_{(w,\zeta)}$ with the same Riemann mapping argument as (z, w, ζ) .

Now, it is easy to verify that for $\delta > 0$, if $(z, w, \zeta) \in M$, $(z, w - \delta, \zeta) \in D$. Let $\phi(w, \zeta, \theta, \delta)$ be the point in $\Gamma_{(w-\delta,\zeta)} \cap (M - \delta)$ whose Riemann mapping argument is θ . By the lemma,

$$\lim_{\delta \rightarrow 0} \int_{D_0 \times \Gamma} \left| \hat{f}(\phi(w, \zeta, \theta, \delta)) - f(R_{(w-\delta,\zeta)}(e^{i\theta}), w - \delta, \zeta) \right|^2 dV = 0,$$

or what is the same

$$\lim_{\delta \rightarrow 0} \int_M \left| \hat{f}(z, w - \delta, \zeta) - f(\psi(z, w - \delta, \zeta)) \right|^2 dV = 0.$$

Now the mapping $(z, w, \zeta) \rightarrow \psi(z, w - \delta, \zeta)$ is a differentiable family of transformations on M , tending to the identity as $\delta \rightarrow 0$. Thus

$$\lim_{\delta \rightarrow 0} \int_M |f(\psi(z, w - \delta, \zeta)) - f(z, w, \zeta)|^2 dV = 0.$$

Thus $\hat{f}(z, w - \delta, \zeta) \rightarrow \hat{f}(z, w, \zeta)$ as $\delta \rightarrow 0$ in L^2 on M , and since $M - \delta \subset D$, $\hat{f}(z, w - \delta, \zeta)$ is holomorphic on M .

BIBLIOGRAPHY

1. J. J. Kohn, *Boundaries of complex manifolds*, Proc. Conf. Complex Analysis, Springer, Berlin, 1965.
2. H. Lewy, *On the local character of the solutions of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables*, Ann. of Math. **64** (1956), 514–522.
3. R. O. Wells, *On the local holomorphic hull of a real submanifold in several complex variables*, Comm. Pure Appl. Math. **19** (1966), 145–165.

BRANDEIS UNIVERSITY