A LOWER BOUND FOR THE FIRST EIGENVALUE OF SECOND ORDER ELLIPTIC OPERATORS

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Let $L$ be the second order operator

$$L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x),$$

where the matrix $(a_{ij})$ is symmetric and positive definite in a bounded domain $D$. The coefficients are assumed to be real and so smooth that the adjoint operator $L^*$ has continuous coefficients on $D$. Let $f$ and $\phi$ be positive functions on $D$. By the methods of Hooker [1] and Protter [2], it can be shown that the lowest eigenvalue $\lambda_1$ of the problem

(1) $Lu + \lambda fu = 0$ in $D$,
(2) $u = 0$ on $\partial D$,

satisfies the inequality

$$\text{Re} \, \lambda_1 \geq \inf_{D} \left(- (L\phi + L^*\phi)/2f\phi \right).$$

If $L$ is selfadjoint, the inequality reduces to the extension of the inequality of Barta

$$\text{Re} \, \lambda_1 \geq \inf_{D} (- L\phi/f\phi),$$

which Protter and Weinberger [3] recently established for nonselfadjoint second order elliptic operators.

The purpose of this paper is to show that the inequality (3) can be generalized to

$$\text{Re} \, \lambda_1 \geq \inf_{D} \left[ - \frac{1}{2f} \left( \frac{L\phi}{\phi} + \frac{L^*\psi}{\psi} \right) \right]$$

for arbitrary positive functions $\phi$ and $\psi$ of class $C^2(D)$. Furthermore, we shall show that (4) also holds under more general boundary conditions, provided additional conditions are imposed on $\phi$ and $\psi$ at the boundary. Such problems were studied by Hooker [1].

Let $g$ and $h$ be nonnegative functions on $\partial D$ such that $g^2 + h^2 > 0$. 

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We shall impose a boundary condition of the form
\begin{equation}
Bu = gu + h \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D,
\end{equation}
where $\frac{\partial u}{\partial \nu}$ denotes the conormal derivative associated with the operator $L$; i.e.,
\[
\frac{\partial}{\partial \nu} = \sum_{i,j} a_{ij} n_i \frac{\partial}{\partial x_j},
\]
where $n_i$ denotes the component of the outward unit normal to $\partial D$ in the direction $x_i$. We shall also employ the adjoint boundary operator defined by
\[
B^* = g + h(k + \partial/\partial \nu),
\]
where
\[
k = \sum_{i,j} \frac{\partial}{\partial x_j} (a_{ij}) n_i - \sum_i b_i n_i.
\]

**Theorem.** Let $\phi$ and $\psi$ be positive functions of class $C^2(\overline{D})$ satisfying
\begin{equation}
(B\phi/\phi) + (B^*\psi/\psi) = 0 \quad \text{on } \partial D.
\end{equation}
Then the lowest eigenvalue of the problem (1), (5) satisfies the inequality (4).

**Proof.** Let $v$ be a positive function of class $C^2(\overline{D})$ and set
\[
I(v, \lambda) = -\frac{1}{2} \int_D [\bar{u}(Lu + \lambda \delta u) + u(L\bar{u} + \lambda \delta \bar{u})] dx.
\]
Several applications of Green's Theorem yield
\begin{equation}
I(v, \lambda) = \int_D \left\{ v \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \bar{u}}{\partial x_j} - \left[ \frac{1}{2} L^* v + \frac{1}{2} \delta v + (\Re \lambda) \delta v \right] |u|^2 \right\} dx
\end{equation}
\begin{equation}
+ \int_{\partial D} \left[ \frac{1}{2} |u|^2 \left( \frac{\partial v}{\partial \nu} + kv \right) - \frac{1}{2} v \frac{\partial}{\partial \nu} |u|^2 \right] dS.
\end{equation}
If $P_i$, $i=1, 2, \cdots, n$, are functions of class $C^1(\overline{D})$, then
\[
\int_D \sum_i \frac{\partial}{\partial x_i} (P_i |u|^2) dx - \int_{\partial D} |u|^2 \sum_i P_i n_i dS = 0.
\]
Adding this to (7), we obtain
\[ I(v, \lambda) = \int_D \left\{ v \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \bar{u}}{\partial x_j} + \sum_i P_i \left( u \frac{\partial \bar{u}}{\partial x_i} + \bar{u} \frac{\partial u}{\partial x_i} \right) - \left[ \frac{1}{2} L^*v + \frac{1}{2} cv + (\text{Re} \, \lambda)f_v - \sum_i \frac{\partial P_i}{\partial x_i} \right] |u|^2 \right\} dx \]

\[ + \int_{\partial D} \left[ \frac{1}{2} |u|^2 \left( \frac{\partial v}{\partial \nu} + kv \right) - \frac{1}{2} v \frac{\partial}{\partial \nu} |u|^2 - |u|^2 \sum_i P_i n_i \right] dS. \]

For \( \phi \) and \( \psi \) positive functions of class \( C^2(\bar{D}) \), an easy computation yields

\[ \frac{1}{2} L^* \left( \frac{\psi}{\phi} \right) + \frac{1}{2} c \frac{\psi}{\phi} = \frac{\psi}{\phi} \left( \frac{1}{2} \frac{L \phi}{\phi} + \frac{1}{2} \frac{L^* \psi}{\psi} \right) - \sum_{i,j} \left[ \frac{\partial}{\partial x_i} \left( \frac{a_{ij} \psi \frac{\partial \phi}{\phi^2}}{\phi^2} \frac{\partial}{\partial x_j} + \frac{a_{ij} \phi \frac{\partial \psi}{\phi^2}}{\phi^3} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \right) \right]. \]

Thus, if we set \( v = \psi / \phi \) and

\[ P_i = -\frac{\psi}{\phi} \sum_j a_{ij} \frac{\partial \phi}{\partial x_j} \]

and make use of the boundary condition (5), the equation (8) assumes the form

\[ I \left( \frac{\psi}{\phi}, \lambda \right) = \int_D \frac{\psi}{\phi} \left\{ \sum_{i,j} a_{ij} \left( \frac{\partial u}{\partial x_i} - u \frac{\partial \phi}{\phi} \right) \left( \frac{\partial \bar{u}}{\partial x_j} - \bar{u} \frac{\partial \phi}{\phi} \right) - \left[ \frac{1}{2} \frac{L \phi}{\phi} + \frac{1}{2} \frac{L^* \psi}{\psi} + (\text{Re} \, \lambda)f \right] \right\} dx \]

\[ + \int_{\Sigma} \frac{\psi}{\phi} \frac{1}{2 \phi h} \left( \frac{B \phi}{\phi} + \frac{B^* \psi}{\psi} \right) dS, \]

where \( \Sigma \) is the subset of \( \partial D \) on which \( h \) is positive. Hence, because of the hypothesis (6), if \( \lambda \) is chosen so that

\[ \frac{1}{2} (L \phi / \phi + L^* \psi / \psi) + (\text{Re} \, \lambda)f < 0, \]

then \( I(\psi / \phi, \lambda) \) is positive for all \( u \) not identically zero. Consequently, \( \lambda \) is not an eigenvalue and the theorem is proved.

For the case of Dirichlet boundary data, we have \( h = 0, g = 1 \), and hence condition (6) is satisfied for arbitrary positive functions \( \phi \) and \( \psi \).

Finally, we note that a generalization of (4) involving a finite num-
ber of pairs of positive functions, $\phi_k$ and $\psi_k$, can be found by considering the sum $\sum I(\psi_k/\phi_k, \lambda)$.

Bibliography


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