A LOWER BOUND FOR THE FIRST EIGENVALUE OF SECOND ORDER ELLIPTIC OPERATORS[1]

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Let L be the second order operator

$$L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x),$$

where the matrix (a_{ij}) is symmetric and positive definite in a bounded domain D. The coefficients are assumed to be real and so smooth that the adjoint operator L^* has continuous coefficients on $\overline{D}$. Let f and ϕ be positive functions on $\overline{D}$. By the methods of Hooker [1] and Protter [2], it can be shown that the lowest eigenvalue λ_1 of the problem

$$Lu + \lambda f u = 0 \quad \text{in } D, \tag{1}$$

$$u = 0 \quad \text{on } \partial D, \tag{2}$$

satisfies the inequality

$$\operatorname{Re} \lambda_1 \geqq \inf_D (-(L\phi + L^*\phi)/2f\phi). \tag{3}$$

If L is selfadjoint, the inequality reduces to the extension of the inequality of Barta

$$\operatorname{Re} \lambda_1 \geqq \inf_D (-L\phi/f\phi),$$

which Protter and Weinberger [3] recently established for nonselfadjoint second order elliptic operators.

The purpose of this paper is to show that the inequality (3) can be generalized to

$$\operatorname{Re} \lambda_1 \geqq \inf_D \left[-\frac{1}{2f} \left(\frac{L\phi}{\phi} + \frac{L^*\psi}{\psi} \right) \right] \tag{4}$$

for arbitrary positive functions ϕ and ψ of class $C^2(\overline{D})$. Furthermore, we shall show that (4) also holds under more general boundary conditions, provided additional conditions are imposed on ϕ and ψ at the boundary. Such problems were studied by Hooker [1].

Let g and h be nonnegative functions on ∂D such that $g^2+h^2>0$.

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We shall impose a boundary condition of the form

$$Bu \equiv gu + h\,\partial u/\partial\nu = 0 \qquad \text{on } \partial D, \tag{5}$$

where $\partial u/\partial\nu$ denotes the conormal derivative associated with the operator L; i.e.,

$$\frac{\partial}{\partial\nu} = \sum_{i,j} a_{ij}n_i \frac{\partial}{\partial x_j},$$

where n_i denotes the component of the outward unit normal to ∂D in the direction x_i. We shall also employ the adjoint boundary operator defined by

$$B^* = g + h(k + \partial/\partial\nu),$$

where

$$k = \sum_{i,j} \frac{\partial}{\partial x_j}(a_{ij})n_i - \sum_i b_i n_i.$$

THEOREM. *Let ϕ and ψ be positive functions of class $C^2(\overline{D})$ satisfying*

$$(B\phi/\phi) + (B^*\psi/\psi) \geqq 0 \quad \textit{on } \partial D. \tag{6}$$

Then the lowest eigenvalue of the problem (1), (5) *satisfies the inequality* (4).

PROOF. Let v be a positive function of class $C^2(\overline{D})$ and set

$$I(v,\lambda) = -\tfrac{1}{2}\int_D v[\bar{u}(Lu + \lambda fu) + u(L\bar{u} + \bar{\lambda}f\bar{u})]dx.$$

Several applications of Green's Theorem yield

$$\begin{aligned} I(v,\lambda) &= \int_D \left\{ v\sum_{i,j} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \bar{u}}{\partial x_j} - \left[\frac{1}{2}L^*v + \frac{1}{2}cv + (\operatorname{Re}\lambda)fv\right]|u|^2\right\} dx \\ &\quad + \int_{\partial D}\left[\frac{1}{2}|u|^2\left(\frac{\partial v}{\partial\nu} + kv\right) - \frac{1}{2}v\frac{\partial}{\partial\nu}|u|^2\right]dS. \end{aligned} \tag{7}$$

If P_i, $i = 1, 2, \cdots, n$, are functions of class $C^1(\overline{D})$, then

$$\int_D \sum_i \frac{\partial}{\partial x_i}(P_i|u|^2)dx - \int_{\partial D}|u|^2\sum_i P_i n_i dS = 0.$$

Adding this to (7), we obtain

$$
\begin{aligned}
I(v,\lambda) = \int_D \Bigg\{ & v \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \bar{u}}{\partial x_j} + \sum_i P_i \left(u \frac{\partial \bar{u}}{\partial x_i} + \bar{u} \frac{\partial u}{\partial x_i} \right) \\
& - \left[\frac{1}{2} L^* v + \frac{1}{2} cv + (\operatorname{Re} \lambda) f v - \sum_i \frac{\partial P_i}{\partial x_i} \right] |u|^2 \Bigg\} dx \\
& + \int_{\partial D} \left[\frac{1}{2} |u|^2 \left(\frac{\partial v}{\partial \nu} + kv \right) - \frac{1}{2} v \frac{\partial}{\partial \nu} |u|^2 - |u|^2 \sum_i P_i n_i \right] dS.
\end{aligned} \tag{8}
$$

For ϕ and ψ positive functions of class $C^2(\overline{D})$, an easy computation yields

$$
\begin{aligned}
& \frac{1}{2} L^* \left(\frac{\psi}{\phi} \right) + \frac{1}{2} c \frac{\psi}{\phi} \\
& = \frac{\psi}{\phi} \left(\frac{1}{2} \frac{L\phi}{\phi} + \frac{1}{2} \frac{L^*\psi}{\psi} \right) - \sum_{i,j} \left[\frac{\partial}{\partial x_i} \left(\frac{a_{ij}\psi}{\phi^2} \frac{\partial \phi}{\partial x_j} \right) + \frac{a_{ij}\psi}{\phi^3} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \right].
\end{aligned}
$$

Thus, if we set $v = \psi/\phi$ and

$$
P_i = -\frac{\psi}{\phi^2} \sum_j a_{ij} \frac{\partial \phi}{\partial x_j}
$$

and make use of the boundary condition (5), the equation (8) assumes the form

$$
\begin{aligned}
I\left(\frac{\psi}{\phi}, \lambda \right) = \int_D \frac{\psi}{\phi} \Bigg\{ & \sum_{i,j} a_{ij} \left(\frac{\partial u}{\partial x_i} - \frac{u}{\phi} \frac{\partial \phi}{\partial x_i} \right) \left(\frac{\partial \bar{u}}{\partial x_j} - \frac{\bar{u}}{\phi} \frac{\partial \phi}{\partial x_j} \right) \\
& - \left[\frac{1}{2} \frac{L\phi}{\phi} + \frac{1}{2} \frac{L^*\psi}{\psi} + (\operatorname{Re} \lambda) f \right] |u|^2 \Bigg\} dx \\
& + \int_\Sigma \frac{1}{2} \frac{\psi |u|^2}{\phi h} \left(\frac{B\phi}{\phi} + \frac{B^*\psi}{\psi} \right) dS,
\end{aligned}
$$

where Σ is the subset of ∂D on which h is positive. Hence, because of the hypothesis (6), if λ is chosen so that

$$
\tfrac{1}{2}(L\phi/\phi + L^*\psi/\psi) + (\operatorname{Re} \lambda) f < 0,
$$

then $I(\psi/\phi, \lambda)$ is positive for all u not identically zero. Consequently, λ is not an eigenvalue and the theorem is proved.

For the case of Dirichlet boundary data, we have $h = 0$, $g = 1$, and hence condition (6) is satisfied for arbitrary positive functions ϕ and ψ.

Finally, we note that a generalization of (4) involving a finite num-

ber of pairs of positive functions, ϕ_k and ψ_k, can be found by considering the sum $\sum I(\psi_k/\phi_k,\lambda)$.

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