ON THE TRANSFORM OF A SINGULAR OR AN ABSOLUTELY CONTINUOUS MEASURE

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Let \( G \) be a locally compact abelian group with dual \( \Gamma \). A well-known theorem of Bochner [1], generalized by Eberlein [2], is the following:

"A continuous function \( \varphi \) defined on \( \Gamma \) is the Fourier-Stieltjes transform of a finite (regular) measure on \( G \) if and only if there is a constant \( A \) such that for every trigonometric polynomial

\[
p(x) = \sum c_i(-x, \gamma_i), \quad \gamma_i \in \Gamma,
\]

the relation \( \|p\| \leq 1 \) implies \( |\sum c_i \varphi(\gamma_i)| \leq A.\)"

If we take \( A = \sup |\sum c_i \varphi(\gamma_i)| \) where the sup is over all polynomials \( p \) for which \( \|p\| \leq 1 \), then whatever be \( \epsilon > 0 \), there is a polynomial

\[
p(x) = \sum c_i(-x, \gamma_i), \quad \gamma_i \in \Gamma,
\]
such that \( \|p\| \leq 1 \) and \( |\sum c_i \varphi(\gamma_i)| > A - \epsilon.\)

A splitting of the last property gives very simple (mutually exclusive) characterizations of the transform of a singular or an absolutely continuous measure. We prove the following theorems:

THEOREM 1. A continuous function \( \varphi \) defined on \( \Gamma \) is the Fourier-Stieltjes transform of a singular measure on \( G \) if and only if there is a constant \( A \) such that

(i) for every trigonometric polynomial \( p(x) = \sum c_i(-x, \gamma_i), \gamma_i \in \Gamma \), the relation \( \|p\| \leq 1 \) implies \( |\sum c_i \varphi(\gamma_i)| \leq A; \)

(ii) whatever be \( \epsilon > 0 \) and the compact set \( K \) in \( \Gamma \) there is a polynomial

\[
p(x) = \sum c_i(-x, \gamma_i), \quad \gamma_i \in \Gamma, \gamma_i \in K,
\]
such that \( \|p\| \leq 1 \) and \( |\sum c_i \varphi(\gamma_i)| > A - \epsilon.\)

THEOREM 2. A continuous function \( \varphi \) defined on \( \Gamma \) is the transform of an absolutely continuous measure on \( G \) if and only if

(i) there is a constant \( A \) such that for every polynomial

\[
p(x) = \sum c_i(-x, \gamma_i), \quad \gamma_i \in \Gamma,
\]

the relation \( \|p\| \leq 1 \) implies \( |\sum c_i \varphi(\gamma_i)| \leq A; \)

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(iia) whatever be \( \epsilon > 0 \) there is a compact set \( K \) in \( \Gamma \) such that for every polynomial \( p(x) = \sum c_i(-x, \gamma_i) \), \( \gamma_i \in \Gamma \), \( \gamma_i \in K \), the relation \( \|p\|_\infty \leq 1 \) implies \( \left| \sum c_i \phi(\gamma_i) \right| < \epsilon \).

**Proof of Theorem 1.**

**Necessity of (iia).** Let \( \varphi(\gamma) = \mu_\epsilon(\gamma) \) where \( \mu_\epsilon \) is singular and \( \|\mu_\epsilon\| = A \). Let \( \epsilon > 0 \) and the compact set \( K \) in \( \Gamma \) be given. There is a polynomial

\[
r(x) = \sum_{j=1}^N b_j(-x, \gamma_j'), \quad \gamma_j' \in \Gamma,
\]

such that \( \|r\|_\infty \leq 1 \) and \( \left| \sum b_j \mu_\epsilon(\gamma_j') \right| = \left| \int \sigma r(x) d\mu_\epsilon(x) \right| > A - \epsilon \).

The set \( C = \{-\gamma_1', \ldots, -\gamma_N'\} \cup -K \) being compact, there is a \( \hat{k} \in L^1(G) \) such that \( \hat{k}(\gamma) = 1 \) on \( C \); i.e., \( \hat{k}(-\gamma_j') = 1, j = 1, \ldots, N \), and \( \hat{k}(-\gamma) = 1 \) for \( \gamma \in K \) and such that \( \|\hat{k}\|_1 < 1 + \epsilon \) (see, e.g., [3, p. 53]). Put \( k'(x) = \hat{k}(-x) \) and \( f(x) = (k' \ast \mu_\epsilon)(x) \). Then \( f \in L^1(G) \) and \( \hat{f}(\gamma) = \hat{k}(-\gamma) \mu_\epsilon(\gamma) \) for \( \gamma \in \Gamma \). Also

\[
\left| \int_G r(x) f(x) dx \right| = \left| \sum b_j \hat{f}(\gamma_j') \right| = \left| \sum b_j \hat{k}(-\gamma_j') \mu_\epsilon(\gamma_j') \right|
\]

\[
= \left| \sum b_j \mu_\epsilon(\gamma_j') \right| > A - \epsilon.
\]

Hence \( \|f\|_1 > A - \epsilon \).

Now consider the measure \( d\mu = d\mu_\epsilon - f(x) dx \). We have \( \|\mu\| = \|\mu_\epsilon\| + \|f\|_1 > 2A - \epsilon \). Hence there exists a polynomial \( q(x) = \sum d_i(-x, \gamma_i) \) such that \( \|q\|_\infty \leq 1 \) and \( \left| \sum d_i \mu_\epsilon(\gamma_i) \right| > 2A - \epsilon \). Put \( p(x) = \frac{1}{2} [q(x) - (q \ast \hat{k})(x)] \). Then

\[
(1) \quad p(x) = \sum c_i(-x, \gamma_i)
\]

where \( c_i = \frac{1}{2} [d_i - d_i \hat{k}(-\gamma_i)] \) so that

\[
(2) \quad c_i = 0 \quad \text{for} \quad \gamma_i \in K.
\]

Also,

\[
(3) \quad \|p\|_\infty \leq \frac{1}{2} \left( 1 + \frac{1}{2} (1 + \epsilon) \right) = 1 + \epsilon/2.
\]

Now

\[
\sum c_i \mu_\epsilon(\gamma_i) = \frac{1}{2} \sum [d_i \mu_\epsilon(\gamma_i) - d_i \mu_\epsilon(\gamma_i) \hat{k}(-\gamma_i)]
\]

\[
= \frac{1}{2} \sum d_i [\mu_\epsilon(\gamma_i) - \hat{f}(\gamma_i)] = \frac{1}{2} \sum d_i \mu_\epsilon(\gamma_i).
\]

Hence

\[
(4) \quad \left| \sum c_i \mu_\epsilon(\gamma_i) \right| > A - \epsilon/2.
\]

Since \( \epsilon > 0 \) is arbitrary, relations (1) and (2), (3) and (4) prove the necessity of (iia).
Sufficiency of (i) and (iis). We know already by Bochner’s theorem that \( \varphi \) is the transform of a regular finite measure \( \mu \), where \( \|\mu\| = A \). We show that \( \mu \) is singular. Let \( d\mu = d\mu_s + g(x)dx \) be the Lebesgue decomposition of \( \mu \). \( \epsilon > 0 \) being given, there is an \( h \in L^1(G) \), whose transform \( \hat{h} \) vanishes outside some compact set \( K \), and such that \( \|g - h\|_1 < \epsilon \). Put \( dv = dv - h(x)dx = d\mu_s + (g - h)dx \). Then \( \vartheta(\gamma) \) coincides with \( \varphi(\gamma) \) outside \( K \). Put \( A_s = \|\mu_s\| \). By (iis) there is a polynomial

\[
p(x) = \sum c_i(x, \gamma_i), \quad \gamma_i \in K,
\]
such that \( \|p\|_\infty \leq 1 \) and \( \|\sum c_i(g(\gamma_i) - \hat{h}(\gamma_i)) + \sum c_i\mu_s(\gamma_i)\| > A - \epsilon \). The l.h.s. is at most \( A_s + \epsilon \). Therefore, \( A_s + \epsilon \geq A - \epsilon \), i.e., \( A_s \geq A - 2\epsilon \). Since \( A_s \leq A \) we conclude \( A_s = A \) and therefore \( \|g\|_1 = 0 \). Thus \( \mu \) is singular and the proof is complete.

Proof of Theorem 2.

Necessity of (iia). Let \( \varphi \) be the transform \( \hat{f} \) of some \( f \in L^1(G) \) and let \( \epsilon > 0 \) be given. There is a \( k \in L^1(G) \) with compact support \( K \), such that \( \|f - f * k\|_1 < \epsilon \). If now \( p(x) = \sum c_i(-x, \gamma_i), \gamma_i \in \Gamma, \gamma_i \in K \) with \( \|p\|_\infty \leq 1 \), then, for \( p'(x) = p(-x) \),

\[
\left| \sum c_i \varphi(\gamma_i) \right| = \left| \sum c_i \hat{f}(\gamma_i) \right| = \left| \sum c_i(\hat{f}(\gamma_i) - \hat{k}(\gamma_i)\hat{f}(\gamma_i)) \right| = \left| (\hat{p}' * (f - k * f))(0) \right| \leq \|\hat{p}'\|_\infty \|f - k \ast f\|_1 < \epsilon.
\]

This proves the necessity of (iia).

Sufficiency of (i) and (iia). We know already that \( \varphi \) is the transform \( \hat{\mu} \) of some finite measure on \( G \). Let \( d\mu = d\mu_s + f(x)dx \) be the Lebesgue decomposition of \( \mu \). Put \( A_s = \|\mu_s\| \). Let \( \epsilon > 0 \) be given. Let \( K_1 \) be the compact set in \( \Gamma \) associated to \( \epsilon \) by (iia), and let \( K_2 \) be the compact set associated to \( \epsilon \) and the absolutely continuous measure \( f(x)dx \) by the necessity just proved. Put \( K = K_1 \cup K_2 \). By Theorem 1 there is a polynomial \( p(x) = \sum c_i(-x, \gamma_i), \gamma_i \in \Gamma, \gamma_i \in K \), such that \( \|p\|_\infty \leq 1 \) and \( \|\sum c_i \mu_s(\gamma_i)\| > A_s - \epsilon \). Also, \( \|\sum c_i \varphi(\gamma_i)\| < \epsilon \) since \( K_1 \subset K \) and \( \|\sum c_i \hat{f}(\gamma_i)\| < \epsilon \) since \( K_2 \subset K \). We conclude

\[
A_s - \epsilon < \left| \sum c_i \mu_s(\gamma_i) \right| = \left| \sum c_i \varphi(\gamma_i) - \hat{f}(\gamma_i) \right| < 2\epsilon,
\]
i.e., \( A_s < 3\epsilon \). Since \( \epsilon \) is arbitrary, we conclude \( A_s = 0 \) so that \( d\mu_s = 0 \) and \( \mu \) is absolutely continuous. This completes the proof of the theorem.

References


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