M-SIMILARITY AND ISOMORPHISMS IN B_0-SPACES[1]

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Let X be a B_0-space, i.e. a locally convex Frechét space. A system $(x_i; f_i)$ with $(x_i) \subset X$ and $(f_i) \subset X^*$ is a *generalized basis* for X if (a) $f_i(x_j) = \delta_{ij}$ for all i and j, and (b) $f_i(x) = 0$ for all i implies $x = 0$ [2]. The sequence space $F(X) = \{(f_i(x)) : x \in X\}$ becomes a B_0-space equivalent to X when equipped with the topology induced by the map $F(\cdot) = (f_i(\cdot))$. Arsove and Edwards proved that a space X, with generalized basis $(x_i; f_i)$, is isomorphic (i.e., linearly homomorphic) to a space Y if and only if Y has a generalized basis $(y_i; g_i)$ such that, as sets, $F(X) = G(Y)$. The author discussed [3] a concept dual to that of generalized basis, and obtained isomorphism theorems analogous to the Arsove-Edwards theorem. W. Ruckle [6] discussed matrix maps which preserve bases in B_0-spaces. (A system $(x_i; f_i)$ is a basis for X if it is a generalized basis, and $x = \sum f_i(x) x_i$ for every x in X.)

Here we introduce a generalized notion of similarity and obtain results which contain the above mentioned work in the context of B_0-spaces.

DEFINITION. Let $(x_i; f_i)$ be a biorthogonal system for a B_0-space X (i.e., $f_i(x_j) = \delta_{ij}$ for all i, j), and $(y_i; g_i)$ such a system for a B_0-space Y. We shall say that $(x_i; f_i)$ and $(y_i; g_i)$ are *M-similar* if either (a) there exists a matrix $A: F(X) \to G(Y)$ which is one-to-one and onto, or (b) there exists a matrix $B: G(Y) \to F(X)$ which is one-to-one and onto. If U is a B_0-space of sequences, let $U' = \{(t_i) : \sum t_i u_i$ converges for all (u_i) in $U\}$.

THEOREM 1. *If the B_0-spaces X and Y have M-similar generalized bases, $(x_i; f_i)$ and $(y_i; g_i)$, respectively, then X and Y are isomorphic.*

PROOF. By hypothesis, we may assume that there is a matrix $A: F(X) \to G(Y)$ which is one-to-one and onto. Since $F(X)$ and $G(Y)$ are FK-spaces (see, eg. [9, p. 202]), we conclude that A defines an isomorphism.

A converse statement is provided by the Arsove-Edwards theorem, since similar generalized bases are trivially M-similar. The following is a somewhat stronger result.

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THEOREM 2. *Let X and Y be B_0-spaces isomorphic under $T: X \to Y$. If $(x_i; f_i)$ and $(y_i; g_i)$ are generalized bases for X and Y, respectively, they are M-similar if either* (a) $((F^{-1})^*T^*)(g_i)$ *is in $F(X)'$ for each i, or* (b) $((G^{-1})^*(T^{-1})^*)(f_i)$ *is in $G(Y)'$ for each i.*

PROOF. For any sequence α in $F(X)$, we have $(GTF^{-1})(\alpha) = \beta$ in $G(Y)$ with $\beta_j = g_j(TF^{-1})(\alpha)$. Assuming (a), $((F^{-1})^*T^*)(g_j) = (\gamma_{jk}) \in F(X)'$, and $\beta_j = \sum_{k=1}^{\infty} \gamma_{jk}\alpha_k$. The matrix $\Gamma = (\gamma_{jk})$ therefore gives the isomorphism GTF^{-1}.

Ruckle [7] showed that if $(x_i; f_i)$ is a basis for X, then $F(X)' = (F(X))^*$ with the natural representation. From this, we get

COROLLARY 1. *Let X, Y, $(x_i; f_i)$ and $(y_i; g_i)$ be as in Theorem 2. If $(x_i; f_i)$ is a basis for X, then $(x_i; f_i)$ and $(y_i; g_i)$ are M-similar.*

Another immediate consequence of the last theorem is the following:

COROLLARY 2. *Let U and V be isomorphic FK-spaces. The isomorphism $T: U \to V$ has a matrix representation if $T^*(\delta^i) \in U'$ for all i, where $\delta^i(v) = v_i$ for all i, $v \in V$.*

In the special case that $(x_i; f_i)$ is a basis for X, M-similar to a generalized basis $(y_i; g_i)$ for Y, it is easy to get the analog of Ruckle's theorem [6, p. 548]. Let $S_Y^0 = \{(t_i): \sum t_j y_j \text{ converges in } Y\}$. Using Theorem 1 and Ruckle's proof, we get the following

PROPOSITION 1. *If $(x_i; f_i)$ is a basis for the B_0-space X, and if $(y_i; g_i)$ is a generalized basis, for the B_0-space Y, which is M-similar to $(x_i; f_i)$, then $(y_i; g_i)$ is a basis for Y if and only if $A\alpha \in S_Y^0$ for every $\alpha \in F(X)$, Here A is the matrix guaranteed by M-similarity and Corollary* 1.

It is worth noting that M-similarity does not preserve fundamental sequence: All B_0-spaces have nontotal generalized bases (see, e.g., [3]), and if $(x_i; f_i)$ is a basis for X, $(y_i; g_i)$ a nontotal generalized basis for Y, Corollary 1 tells us that the systems are M-similar. Let $(x_i; f_i)$ be a biorthogonal system for X, and let $\hat{x}_i$ be the canonical image of x_i in X^{**}. We call $(x_i; f_i)$ a *dual generalized basis* for X if $(f_i; \hat{x}_i)$ is a generalized basis for X^* [3]. We shall call biorthogonal systems $(x_i; f_i)$ and $(y_i; g_i)$ *M^*-similar* if $(f_i; \hat{x}_i)$ and $(g_i; \hat{y}_i)$ are M-similar. With this definition, Theorem 1, and the proof of Theorem 2 in [3] we get

THEOREM 3. *Let $(x_i; f_i)$ and $(y_i; g_i)$ be M^*-similar dual generalized bases for the B_0-spaces X and Y, respectively. Then X and Y are isomorphic.*

It is clear that the analog to Theorem 2, above, holds for M^*-similarity.

A biorthogonal system which is both a generalized basis and a dual generalized basis is called a *Markuševič basis*. A Markuševič basis exists in any separable linear topological space with a total sequence of continuous linear functionals [4], [8]. As noted above, a Markuševič basis can be M-similar to a nontotal generalized basis. Following [3], however, we easily obtain the following:

PROPOSITION 2. *If $(x_i; f_i)$ is a Markuševič basis for X which is simultaneously M-similar and M^*-similar to the generalized basis $(y_i; g_i)$ (for Y), then $(y_i; g_i)$ is a Markuševič basis for Y.*

There is a partial converse to this statement. If $A: U \to V$ is one to one and onto, where U and V are FK-spaces, then the rows of A are in U' and the columns of A are in V. With this fact, and the proof of Theorem 2, the following proposition is immediate.

PROPOSITION 3. *Let $(x_i; f_i)$ and $(y_i; g_i)$ be M-similar Markuševič bases (with matrix $A: F(X) \to G(Y)$). The systems are M^*-similar if $G(Y) \subset \tilde{Y}\ (Y^*)'$, where $\tilde{Y}\ (Y^*) = \{(g(y_i)): g \in Y^*\}$.*

It is easy to see that M-similar systems are similar if the matrix involved is the identity matrix. Paley-Wiener theorems (see, e.g., [1]) suggest that M-similar systems are similar for a large class of matrices. The author knows no nontrivial characterization of this class.

Other problems arise in connection with M-similarity since the relation is not transitive. To see this, let $A: U \to V$, with U and V FK-spaces, be a matrix isomorphism with no inverse; that is, there is no matrix $B: V \to U$ with $B(Au) = u$ for all $u \in U$ (see [9, p. 228]). If $C: W \to V$ has the same property, the sequences $(\delta_{ij})_{j=1}^{\infty} = e_i$ are M-similar in U and V and in V and W, but not necessarily in U and W.

REFERENCES

1. M. G. Arsove, *The Paley-Wiener theorem in metric linear spaces*, Pacific J. Math. **10** (1960), 365–379.

2. M. G. Arsove and R. E. Edwards, *Generalized bases in topological linear spaces*, Studia Math. **19** (1960), 95–113.

3. W. J. Davis, *Dual generalized bases in linear topological spaces*, Proc. Amer. Math. Soc. **17** (1966), 1057–1063.

4. V. F. Gaposhkin and M. I. Kadec, *Operator bases in Banach spaces*, Mat. Sb. **61(103)** (1965), 3–12.

5. V. L. Klee, *On the Borelian and projective types of linear subspaces*, Math. Scand. 6 (1958), 189–199.

6. W. Ruckle, *Infinite matrices which preserve Schauder bases*, Duke Math. J. **33** (1966), 541–550.

7. ———, *On the characterization of sequence spaces associated with Schauder bases*, Studia Math. **28** (1967), 279–288.

8. ———, *On the existence of Markouchevitch bases*, (to appear).

9. A. Wilansky, *Functional analysis*, Blaisdell, New York, 1964.

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