ADDITIVE FUNCTIONALS ON $C(Y)$[1]

J. KUELBS

1. **Introduction.** In [1] and [2] Cameron and Graves provide an interesting characterization of the class of additive Wiener measurable functionals on the space C of real valued continuous functions on $[0, 1]$ which vanish at zero. The purpose of this paper is to obtain a similar characterization for additive measurable functionals on $C(Y)$ when Y is the product space $\prod_{k=1}^{\infty}[a_k, b_k]$. It is assumed that Y has the product topology, that $b_k - a_k = O(2^{-k})$, and that $C(Y)$ denotes the real continuous functions on Y with the uniform topology. The measure on $C(Y)$ is the Gaussian measure m defined in [3].

Since the class of open subsets of $C(Y)$ is m-measurable it follows that continuous functionals are measurable and, in particular, that bounded linear functionals on $C(Y)$ will have our representation. In Theorems 2 and 3 the relationship between the Riesz representation and our representation for bounded linear functionals is determined. In §4 it is shown that certain additive functionals on $C(X)$, where X is a compact metric space, also have our representation.

2. **Preliminary results.** Let $Y_n = \prod_{k=1}^{n} [a_k, b_k] \times \alpha_n$ where $\alpha_n = (a_{n+1}, a_{n+2}, \cdots)$ for $n = 1, 2, \cdots$, and by S_n denote the 2^n subsets of Y_n formed by selecting $n-k$ of the first n coordinates and setting each such x_j equal to a_j while the remaining k coordinates among the first n are allowed to vary as they do in Y_n. The symbol S denotes $\bigcup_{n=1}^{\infty} S_n$. If $I \in S$ and I has $k > 0$ coordinates which vary, then μ_I denotes Lebesgue measure on I when I is considered as k-dimensional. If I is the single point $(a_1, a_2, \cdots)$, then μ_I is the measure obtained by placing mass one at this point.

If B is a Borel subset of Y, we define $\nu(B) = \sum_{I \in S} \mu_I(B \cap I)$. Then ν is sigma-additive on the Borel sets and $C(Y)$ is dense in $\mathcal{L}_2(Y)$ with respect to mean square convergence. In fact, polynomials in finitely many coordinates of Y with rational coefficients are dense in $\mathcal{L}_2(Y)$, and, as a result, a countable orthonormal basis of polynomials exists for $\mathcal{L}_2(Y)$.

Let $\{\phi_k(p)\}$ be a complete orthonormal set in $\mathcal{L}_2(Y)$ each of which is a polynomial in a finite number of variables. If $g \in \mathcal{L}_2(Y)$ and $g_n(p) = \sum_{k=1}^{n} c_k \phi_k(p)$ where $c_k = \int_Y g \phi_k d\nu$, then, as is shown in [3], the P.W.Z. (Paley, Wiener, Zygmund) integral $\int_Y g(df)^{\sim} = \lim_n \int_Y g_n df$

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exists for almost all f in $C(Y)$. Here by $\int_Y g_n df$ we mean $\sum_{I\in S}\int_I g_n df$ where $\int_I g_n df$ denotes the ordinary Riemann-Stieltjes integral of g_n with respect to f over I when g_n and f are thought of as functions on I. The existence of $\int_Y g_n df$ for almost all f in $C(Y)$ and the fact that $\int_Y g_n df$ is a Gaussian functional with mean zero and variance $\frac{1}{2}\int_Y g_n^2 d\nu$ is assured by results in [3]. It is also shown in [3] that $\int_Y g(df)^{\sim}$ is a Gaussian functional with expectation zero and variance $\frac{1}{2}\int_Y g^2 d\nu$ which is independent of the complete orthonormal set $\{\phi_k(p)\}$ provided each ϕ_k is a polynomial in a finite number of variables. Consequently, if $g_1, \cdots, g_n$ are orthonormal elements of $\mathcal{L}_2(Y)$, then $\int_Y g_1(df)^{\sim}, \cdots, \int_Y g_n(df)^{\sim}$ form an independent family of Gaussian functionals with mean zero and variance one-half.

In [4] Cameron and Martin introduced a complete orthonormal set of functionals in $\mathcal{L}_2(C)$ where C is as in the Introduction and the measure on C is Wiener measure. In a similar fashion one can introduce a complete orthonormal set of functionals for the space of square integrable functionals $\mathcal{L}_2$ on $C(Y)$.

Let $H_n(u)$ be the partially normalized Hermite polynomial

$$(-1)^n 2^{-n/2}(n!)^{-1/2}e^{u^2}\frac{d^n}{du^n}(e^{-u^2}) \quad \text{for } n = 0, 1, 2, \cdots.$$

If $\{\phi_p(p)\}$ is any orthonormal basis for $\mathcal{L}_2(Y)$, we define

$$\Phi_{m,k}(f) = H_m\left(\int_Y \phi_k(df)^{\sim}\right), \quad m = 0, 1, \cdots : k = 1, 2, \cdots, \tag{2.1}$$

$$\Psi_{m_1,\cdots,m_k}(f) = \Phi_{m_1,1}(f)\cdots\Phi_{m_k,k}(f). \tag{2.2}$$

Since $H_0(u)=1$ it follows that $\Phi_{0,k}(f)=1$ and $\Psi_{m_1,\cdots,m_k,0,\cdots,0}(f) =\Psi_{m_1,\cdots,m_k}(f)$. By $\{\Psi_\alpha\}$ we will denote the set of functionals of the form (2.2) where α is any finite sequence of nonnegative integers. Then $\{\Psi_\alpha\}$ is a complete orthonormal subset for $\mathcal{L}_2$. That is, if F is in $\mathcal{L}_2$, then

$$\lim_{N\to\infty} E\left[F(f) - \sum_{m_1,\ldots,m_N=0}^{N} A_{m_1,\cdots,m_N}\Psi_{m_1,\cdots,m_N}(f)\right]^2 = 0$$

where $A_{m_1,\cdots,m_N}=E[F(f)\Psi_{m_1,\cdots,m_N}(f)]$ and by $E(\cdot)$ we mean integration with respect to the measure m on $C(Y)$. The proof of this can be carried out with only a few minor modifications to the proof of the corresponding result in [4].

Henceforth we will assume $\{\phi_k(p)\}$ is a complete orthonormal set for $\mathcal{L}_2(Y)$ and that each is a polynomial in a finite number of variables. Using the Fourier-Hermite expansion for functionals in $\mathcal{L}_2$ mentioned above and the translation theorem of [3] for the measure m on $C(Y)$ it is possible by application of the techniques of Cameron and Graves found in [1] and [2] to prove the following lemma.

LEMMA 1. *Let $F(f)$ be measurable and additive on $C(Y)$. Let $\{\phi_k(p)\}$ be the complete orthonormal set of polynomials for $\mathfrak{L}_2(Y)$ used in the definition of the Fourier-Hermite functionals and set $\theta_k(p)=\int_{Y(p)}\phi_k dv$ where $Y(p)=\prod_{k=1}^{\infty}[a_k, x_k]$ if $p=(x_1, x_2 \cdots)$. Then $\sum_{k=1}^{\infty}[F(\theta_k)]^2 < \infty$ and, for almost all f in $C(Y)$,*

$$F(f) = \sum_{k=1}^{\infty} F(\theta_k) \int_Y \phi_k df. \tag{2.3}$$

3. **The representation theorem and its relationship to the integral representation of Riesz.** A functional F will be called essentially additive, homogeneous, or linear if it is almost everywhere equal to a functional which is additive, homogeneous, or linear, respectively.

THEOREM 1. *A measurable function F is essentially additive on $C(Y)$ iff for almost all f in $C(Y)$*

$$F(f) = \int_Y h(df)^{\sim} \tag{3.1}$$

where h is in $\mathfrak{L}_2(Y)$.

PROOF. Since F is measurable and essentially additive there exists an additive measurable functional G on $C(Y)$ such that $F(f)=G(f)$ almost everywhere. Now by (2.3)

$$G(f) = \sum_{k=1}^{\infty} G(\theta_k) \int_Y \phi_k df$$

almost everywhere and $\sum_{k=1}^{\infty}[G(\theta_k)]^2<\infty$. Let

$$h = \underset{N}{\text{l.i.m.}} \sum_{k=1}^{N} G(\theta_k)\phi_k,$$

then by definition $\int_Y h(df)^{\sim}=\lim_N \int_Y g_N df$ where $g_N=\sum_{k=1}^{N} c_k\phi_k$ and $c_k=\int_Y\phi_k h dv=G(\theta_k)$. That is, $\int_Y h(df)^{\sim}=\sum_{k=1}^{\infty} G(\theta_k)\int_Y\phi_k df=G(f)$ for almost all f and hence (3.1) holds for F almost surely.

Suppose $F(f)=\int_Y h(df)^{\sim}$ almost surely for some h in $\mathfrak{L}_2(Y)$. Since $\int_Y h(df)^{\sim}$ is linear on a linear subspace of $C(Y)$ of measure one it follows that $\int_Y h(df)^{\sim}$ can be extended to be linear on all of $C(Y)$. Hence $F(f)$ is essentially linear on $C(Y)$ and the theorem is proved.

COROLLARY. *If F is measurable and essentially additive on $C(Y)$,* *then F is essentially linear and has a Gaussian distribution with mean zero and variance $\frac{1}{2}\int_Y h^2 d\nu$ when F is as in* (3.1).

PROOF. The fact that F is essentially linear appears in the proof of Theorem 1. To see that $F(f)$ has the indicated Gaussian distribution simply observe that $F(f)=\int_Y h(df)^\sim$ almost everywhere for some $h\in\mathfrak{L}_2(Y)$. Now $\int_Y h(df)^\sim=\lim_n \int_Y g_n df$ where $g_n=\sum_{k=1}^n c_k\phi_k$ and $c_k=\int_Y h\phi_k d\nu$, and since $\{c_k\int_Y\phi_k df\}$ is a sequence of independent Gaussian functionals with mean zero and variance $c_k^2/2$ the result follows.

If $F(f)$ is a bounded linear functional on $C(Y)$, then the Riesz representation theorem asserts that there exists a finite signed measure μ on the Borel subsets $\mathfrak{B}$ of Y such that $F(f)=\int_Y f d\mu$ for all $f\in C(Y)$. On the other hand, $F(f)=\int_Y h(df)^\sim$ for almost all f in $C(Y)$ where h is in $\mathfrak{L}_2(Y)$. We now proceed to relate the measure μ and the function h.

If μ is a finite signed measure on $\mathfrak{B}$ and τ is a continuous function of Y into Y, then for every $B\in\mathfrak{B}$ we have $\tau^{-1}(B)\in\mathfrak{B}$ and we define μ^τ to be the measure on $\mathfrak{B}$ such that $\mu^\tau(B)=\mu(\tau^{-1}(B))$ for $B\in\mathfrak{B}$.

By τ_n we mean the projection of Y onto Y_n for $n=1, 2, \cdots$. That is, $\tau_n(x_1, \cdots, x_n, x_{n+1}, \cdots)=(x_1, \cdots, x_n, a_{n+1}, \cdots)$ for all p in Y. Now τ_n is continuous and hence $\mu_n=\mu^{\tau_n}$ is a finite signed measure on Y concentrated in Y_n.

LEMMA 2. *If f is in $C(Y)$, then*

$$\lim_{n\to\infty}\int_Y f d\mu_n=\int_Y f d\mu.$$

PROOF. First observe that

$$\int_Y f(p)d\mu_n=\int_Y f(p)d\mu^{\tau_n}=\int_Y f(\tau_n(p))d\mu \quad \text{for } n=1,2,\cdots.$$

Then since $f\in C(Y)$ and τ_n converges uniformly to the identity map, we have

$$\lim_{n\to\infty}\int_Y f(p)d\mu_n=\int_Y f(p)d\mu.$$

Since μ_n is a finite signed measure concentrated in Y_n it follows from [5, p. 288] that there exists a unique function H_n on Y_n such that

(1) H_n is bounded and of bounded variation on all $I\in S_n$,
(2) $H_n(p)=0$ if any $x_k=b_k$ when $p=(x_1, \cdots, x_n, a_{n+1}, \cdots)$,
(3) H_n is left continuous on Y_n except possibly for points p

$=(x_1, \cdots, x_n, a_{n+1}, \cdots)$ when some $x_k = b_k$, and

(4) $\int_{Y_n} f dH_n = \int_Y f d\mu_n$ for all $f \in C(Y)$.

On the other hand, the existence of an H_n satisfying (1), (2), and (3) implies the existence of a measure μ_n concentrated on the Borel subsets of Y_n such that (4) holds. We extend H_n to be zero on $Y - Y_n$. Since $H_n = 0$ on $Y - Y_n$ it follows that

$$\int_Y (-1)^n H_n df = \sum_{I \in S} \int_I (-1)^n H_n df = \sum_{I \in S_n} \int_I (-1)^n H_n df.$$

Now by the integration by parts formula given in [6, p. 415] and the fact that $H_n(p) = 0$ for all $p = (x_1, \cdots, x_n, a_{n+1}, \cdots)$ when some $x_k = b_k$ for $k = 1, \cdots, n$, we obtain $\int_Y (-1)^n H_n df = \int_{Y_n} f dH_n$. It is now possible to relate the Riesz representation of a bounded linear functional and the representation provided in (3.1). In the next theorem we assume $\mu_n = \mu^{\tau_n}$ and that $H_n(p)$ is related to μ_n as above.

Theorem 2. *If F is a bounded linear functional on $C(Y)$ such that $F(f) = \int_Y f d\mu$ where μ is a finite signed measure on $\mathcal{B}$, then $F(f) = \int_Y h(df)^\sim$ for almost all f in $C(Y)$ where $h \in \mathcal{L}_2(Y)$ and $h(p) = \mathrm{l.i.m.}_n (-1)^n H_n(p)$.*

Proof. Since $\lim_n \int_{Y_n} f d\mu_n = \int_Y f d\mu$ and $\int_{Y_n} f d\mu_n = \int_{Y_n} f dH_n = \int_Y (-1)^n H_n df$ we have $F(f) = \lim_n \int_Y (-1)^n H_n df$ for all f in $C(Y)$. Now $F(f) = \int_Y h(df)^\sim$ and since $\int_Y (-1)^n H_n df = \int_Y (-1)^n H_n (df)^\sim$ almost surely on $C(Y)$ it follows that $\lim_{n \to \infty} \int_Y [(-1)^n H_n - h](df)^\sim = 0$ for almost all f. However, $\int_Y [(-1)^n H_n - h](df)^\sim$ is a Gaussian functional with mean zero and variance $\int_Y [(-1)^n H_n - h]^2 d\nu$ so we have $\lim_n \int_Y [(-1)^n H_n - h]^2 d\nu = 0$ as was to be proved.

Lemma 3. *If E is an open subset of $C(Y)$, then $m(E) > 0$.*

Proof. Let $I = \{f \in C(Y) : \|f\| \leqq \lambda\}$ where $\|f\|$ is the uniform norm of f and $\lambda > 0$. Let $\{f_k\}$ be a sequence of polynomials on Y each in a finite number of variables such that $\{f_k\}$ is dense in $C(Y)$. Let $I_k = \{f \in C(Y) : \|f - f_k\| \leqq \lambda\}$. Then $I_k - f_k = I$, and if $F(f) = \chi_I(f)$, we have by the translation theorem in [3] that

$$E(F(f)) = E\left\{F(f + f_k) \exp\left[-\int_Y f_k^2 d\nu - 2\int_Y f_k df\right]\right\}.$$

Thus $m(I) = 0$ if and only if $m(I_k) = 0$. However, $C(Y) = \bigcup_{k=1}^\infty I_k$ and if $m(I) = 0$ then $m(C(Y)) = 0$ which is a contradiction. Thus $m(I) > 0$ and $m(I_k) > 0$ for $k = 1, 2, \cdots$. Since E is open there exists $\lambda > 0$ and f_j such that $I_j = \{f \in C(Y) : \|f - f_j\| \leqq \lambda\}$ is a subset of E. Hence $m(E) \geqq m(I_j) > 0$ as was to be proved.

Let $h(p)$ be a function on Y and define $H_n(p)=h(p)$ for $p\in Y_n$ except when some coordinate x_k of p is b_k and zero otherwise. Further, suppose H_n satisfies conditions (1) and (3) and that μ_n is the finite signed measure concentrated on Y_n related to $(-1)^n H_n$. We then say $\{\mu_n\}$ is obtained from h. If μ is a signed measure on Y then $|\mu|(Y)$ denotes the total variation of μ.

THEOREM 3. *If $F(f)=\int_Y h(df)^\sim$ where $h\in \mathfrak{L}_2(Y)$ and $\{\mu_n\}$ is obtained from h such that $|\mu_n|(Y)<M$ for $n=1, 2, \cdots$, then $F(f)$ is essentially a bounded linear functional on $C(Y)$, and for almost all f in $C(Y)$*

$$F(f) = \lim_n \int_Y f d\mu_n.$$

PROOF. Since $\int_Y f d\mu_n=\int_{Y_n}(-1)^n f dH_n=\int_Y H_n df$ it follows that $\lim_n \int_Y f d\mu_n=\lim_n \int_Y H_n df$. Now H_n converges to h in $\mathfrak{L}_2(Y)$, and for almost all $f\in C(Y)$

$$\lim_n\left[\int_Y H_n df - \int_Y h(df)^\sim\right] = \lim_n \int_Y (H_n - h)(df)^\sim = 0$$

since $\lim_n \int_Y (H_n-h)^2 d\nu=0$. Hence $\lim_n \int_Y f d\mu_n=\int_Y h(df)^\sim=F(f)$ for almost all f in $C(Y)$. Since open sets have positive measure there exists a dense set $\{f_k\}$ in $C(Y)$ such that $\lim_n \int_Y f_k d\mu_n=F(f_k)$ for $k=1, 2, \cdots$. Let $f\in C(Y)$. Then

$$\left|\int_Y f d\mu_n - \int_Y f d\mu_m\right| \leqq 2M\|f - f_k\| + \left|\int_Y f_k d\mu_n - \int_Y f_k d\mu_m\right|,$$

so $\lim_n \int_Y f d\mu_n$ exists for all $f\in C(Y)$. If $L(f)=\lim_n \int_Y f d\mu_n$, then $L(f)$ is linear, $L(f)=F(f)$ almost surely, and $L(f)\leqq M\|f\|$. Hence $F(f)$ is an essentially bounded linear functional on $C(Y)$.

4. Since any compact metric space X is homeomorphic to a closed subset of Y it follows that any additive functional on $C(X)$ gives rise to an additive functional on $C(Y)$. That is, let ϕ be a homeomorphism of X into Y and define $\theta(f)=f(\phi(\cdot)), f\in C(Y)$, then θ maps $C(Y)$ onto $C(X)$ and θ is continuous. The measure m^θ is defined on the Borel subsets $\mathfrak{A}$ of $C(X)$ by the equation $m^\theta(A)=m(\theta^{-1}(A))$, $A\in\mathfrak{A}$. Then if G is an essentially additive m^θ-measurable functional on $C(X)$, we have $F(f)=G(\theta(f))$ as an essentially additive measurable functional on $C(Y)$. Using Theorem 1 with ϕ and θ defined as above, we get the following result.

THEOREM 4. *If a functional G on $C(X)$ is essentially additive and m^θ-measurable, then there exists an h in $\mathfrak{L}_2(Y)$ such that $g=\theta(f)$ implies*

$$G(g) = \int_Y h(df)^{\sim}$$

for almost every g in $C(X)$.

In particular, since the open sets of $C(X)$ are m^θ-measurable it follows that the bounded linear functionals on $C(X)$ have the above representation.

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University of Wisconsin