ADDITIVE FUNCTIONALS ON $C(Y)^1$

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1. Introduction. In [1] and [2] Cameron and Graves provide an interesting characterization of the class of additive Wiener measurable functionals on the space $C$ of real valued continuous functions on $[0, 1]$ which vanish at zero. The purpose of this paper is to obtain a similar characterization for additive measurable functionals on $C(Y)$ when $Y$ is the product space $\prod_{k=1}^{\infty} [a_k, b_k]$. It is assumed that $Y$ has the product topology, that $b_k - a_k = O(2^{-j})$, and that $C(Y)$ denotes the real continuous functions on $Y$ with the uniform topology. The measure on $C(Y)$ is the Gaussian measure $m$ defined in [3].

Since the class of open subsets of $C(Y)$ is $m$-measurable it follows that continuous functionals are measurable and, in particular, that bounded linear functionals on $C(Y)$ will have our representation. In Theorems 2 and 3 the relationship between the Riesz representation and our representation for bounded linear functionals is determined. In §4 it is shown that certain additive functionals on $C(X)$, where $X$ is a compact metric space, also have our representation.

2. Preliminary results. Let $Y_n = \prod_{k=1}^{n} [a_k, b_k] \times \alpha_n$ where $\alpha_n = (a_{n+1}, a_{n+2}, \ldots)$ for $n = 1, 2, \ldots$, and by $S_n$ denote the $2^n$ subsets of $Y_n$ formed by selecting $n-k$ of the first $n$ coordinates and setting each such $x_j$ equal to $a_j$ while the remaining $k$ coordinates among the first $n$ are allowed to vary as they do in $Y_n$. The symbol $S$ denotes $\bigcup_{n=1}^{\infty} S_n$. If $I \in S$ and $I$ has $k > 0$ coordinates which vary, then $\mu_I$ denotes Lebesgue measure on $I$ when $I$ is considered as $k$-dimensional. If $I$ is the single point $(a_1, a_2, \ldots)$, then $\mu_I$ is the measure obtained by placing mass one at this point.

If $B$ is a Borel subset of $Y$, we define $\nu(B) = \sum_{I \in S} \mu_I(B \cap I)$. Then $\nu$ is sigma-additive on the Borel sets and $C(Y)$ is dense in $L_2(Y)$ with respect to mean square convergence. In fact, polynomials in finitely many coordinates of $Y$ with rational coefficients are dense in $L_2(Y)$, and, as a result, a countable orthonormal basis of polynomials exists for $L_2(Y)$.

Let $\{\phi_k(p)\}$ be a complete orthonormal set in $L_2(Y)$ each of which is a polynomial in a finite number of variables. If $g \in L_2(Y)$ and $g_n(p) = \sum_{k=1}^{n} c_k \phi_k(p)$ where $c_k = \int_Y g \phi_k dv$, then, as is shown in [3], the P.W.Z. (Paley, Wiener, Zygmund) integral $\int_Y g(df)^- = \lim_n \int_Y g_n df$.

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exists for almost all $f$ in $C(Y)$. Here by $\int_Y g_n(df)$ we mean $\sum_{t \in S} \int g_n(df)$ where $\int g_n(df)$ denotes the ordinary Riemann-Stieltjes integral of $g_n$ with respect to $f$ over $I$ when $g_n$ and $f$ are thought of as functions on $I$. The existence of $\int_Y g_n(df)$ for almost all $f$ in $C(Y)$ and the fact that $\int_Y g_n(df)$ is a Gaussian functional with mean zero and variance $\frac{1}{2} \int_Y g_n^2(dv)$ is assured by results in [3]. It is also shown in [3] that $\int_Y g(df)^\sim$ is a Gaussian functional with expectation zero and variance $\frac{1}{2} \int_Y g^2(dv)$ which is independent of the complete orthonormal set $\{\phi_k(p)\}$ provided each $\phi_k$ is a polynomial in a finite number of variables. Consequently, if $g_1, \ldots, g_n$ are orthonormal elements of $L_2(Y)$, then $\int_Y g_1(df)^\sim, \ldots, \int_Y g_n(df)^\sim$ form an independent family of Gaussian functionals with mean zero and variance one-half.

In [4] Cameron and Martin introduced a complete orthonormal set of functionals in $L_2(C)$ where $C$ is as in the Introduction and the measure on $C$ is Wiener measure. In a similar fashion one can introduce a complete orthonormal set of functionals for the space of square integrable functionals $L_2$ on $C(Y)$.

Let $H_n(u)$ be the partially normalized Hermite polynomial

$$(-1)^n 2^{-n/2} (n!)^{-1/2} e^{-u^2} \frac{d^n}{du^n} (e^{-u^2}) \quad \text{for } n = 0, 1, 2, \ldots .$$

If $\{\phi_p(p)\}$ is any orthonormal basis for $L_2(Y)$, we define

\begin{align*}
(2.1) \quad & \Phi_{m,k}(f) = H_m \left( \int_Y \phi_k(df)^\sim \right), \quad m = 0, 1, \ldots, k = 1, 2, \ldots, \\
(2.2) \quad & \Psi_{m_1, \ldots, m_k}(f) = \Phi_{m_1,1}(f) \cdots \Phi_{m_k,k}(f).
\end{align*}

Since $H_0(u) = 1$ it follows that $\Phi_{0,k}(f) = 1$ and $\Psi_{m_1, \ldots, m_k,0,0,\ldots,0}(f) = \Psi_{m_1, \ldots, m_k}(f)$. By $\{\Psi_\alpha\}$ we will denote the set of functionals of the form (2.2) where $\alpha$ is any finite sequence of nonnegative integers. Then $\{\Psi_\alpha\}$ is a complete orthonormal subset for $L_2$. That is, if $F$ is in $L_2$, then

$$\lim_{N \to \infty} E \left[ F(f) - \sum_{m_1, \ldots, m_N=0} A_{m_1, \ldots, m_N} \Psi_{m_1, \ldots, m_N}(f) \right]^2 = 0$$

where $A_{m_1, \ldots, m_N} = E[F(f)\Psi_{m_1, \ldots, m_N}(f)]$ and by $E(\cdot)$ we mean integration with respect to the measure $m$ on $C(Y)$. The proof of this can be carried out with only a few minor modifications to the proof of the corresponding result in [4].

Henceforth we will assume $\{\phi_k(p)\}$ is a complete orthonormal set for $L_2(Y)$ and that each is a polynomial in a finite number of variables. Using the Fourier-Hermite expansion for functionals in $L_2$ men-
tioned above and the translation theorem of [3] for the measure \( m \) on \( C(Y) \) it is possible by application of the techniques of Cameron and Graves found in [1] and [2] to prove the following lemma.

**Lemma 1.** Let \( F(f) \) be measurable and additive on \( C(Y) \). Let \( \{\phi_k(p)\} \) be the complete orthonormal set of polynomials for \( L_2(Y) \) used in the definition of the Fourier-Hermite functionals and set \( \theta_k(p) = \int_Y (p)\phi_k dv \) where \( Y(p) = \prod_{k=1}^n [a_k, x_k] \) if \( p = (x_1, x_2, \ldots) \). Then \( \sum_{k=1}^\infty |F(\theta_k)|^2 < \infty \) and, for almost all \( f \) in \( C(Y) \),

\[
F(f) = \sum_{k=1}^\infty F(\theta_k) \int_Y \phi_k df.
\]

(2.3)

3. The representation theorem and its relationship to the integral representation of Riesz. A functional \( F \) will be called essentially additive, homogeneous, or linear if it is almost everywhere equal to a functional which is additive, homogeneous, or linear, respectively.

**Theorem 1.** A measurable function \( F \) is essentially additive on \( C(Y) \) iff for almost all \( f \) in \( C(Y) \)

\[
F(f) = \int_Y h(df) ~
\]

where \( h \) is in \( L_2(Y) \).

**Proof.** Since \( F \) is measurable and essentially additive there exists an additive measurable functional \( G \) on \( C(Y) \) such that \( F(f) = G(f) \) almost everywhere. Now by (2.3)

\[
G(f) = \sum_{k=1}^\infty G(\theta_k) \int_Y \phi_k df
\]

almost everywhere and \( \sum_{k=1}^\infty |G(\theta_k)|^2 < \infty \). Let

\[
h = \lim_{N} \sum_{k=1}^{N} G(\theta_k) \phi_k,
\]

then by definition \( \int_Y h(df) ~ = \lim_{N} \int_Y g_N df \) where \( g_N = \sum_{k=1}^{N} c_k \phi_k \) and \( c_k = \int_Y \phi_k dv = G(\theta_k) \). That is, \( \int_Y h(df) ~ = \sum_{k=1}^\infty G(\theta_k) \int_Y \phi_k df = G(f) \) for almost all \( f \) and hence (3.1) holds for \( F \) almost surely.

Suppose \( F(f) = \int_Y h(df) ~ \) almost surely for some \( h \) in \( L_2(Y) \). Since \( \int_Y h(df) ~ \) is linear on a linear subspace of \( C(Y) \) of measure one it follows that \( \int_Y h(df) ~ \) can be extended to be linear on all of \( C(Y) \). Hence \( F(f) \) is essentially linear on \( C(Y) \) and the theorem is proved.

**Corollary.** If \( F \) is measurable and essentially additive on \( C(Y) \),
then $F$ is essentially linear and has a Gaussian distribution with mean zero and variance $\frac{1}{2} \int_Y h^2 dv$ when $F$ is as in (3.1).

**Proof.** The fact that $F$ is essentially linear appears in the proof of Theorem 1. To see that $F(f)$ has the indicated Gaussian distribution simply observe that $F(f) = \int_Y h(df)^\sim$ almost everywhere for some $h \in \mathcal{L}_1(Y)$. Now $\int_Y h(df)^\sim = \lim_n \int_Y g_n df$ where $g_n = \sum_{k=1}^{n} c_k \phi_k$ and $c_k = \int_Y h \phi_k dv$, and since $\{c_k \int_Y \phi_k df\}$ is a sequence of independent Gaussian functionals with mean zero and variance $c_k^2/2$ the result follows.

If $F(f)$ is a bounded linear functional on $C(Y)$, then the Riesz representation theorem asserts that there exists a finite signed measure $\mu$ on the Borel subsets $\mathcal{B}$ of $Y$ such that $F(f) = \int_Y f d\mu$ for all $f \in C(Y)$. On the other hand, $F(f) = \int_Y h(df)^\sim$ for almost all $f$ in $C(Y)$ where $h$ is in $\mathcal{L}_1(Y)$. We now proceed to relate the measure $\mu$ and the function $h$.

If $\mu$ is a finite signed measure on $\mathcal{B}$ and $\tau$ is a continuous function of $Y$ into $Y$, then for every $B \in \mathcal{B}$ we have $\tau^{-1}(B) \in \mathcal{B}$ and we define $\mu^\tau$ to be the measure on $\mathcal{B}$ such that $\mu^\tau(B) = \mu(\tau^{-1}(B))$ for $B \in \mathcal{B}$.

By $\tau_n$ we mean the projection of $Y$ onto $Y_n$ for $n = 1, 2, \ldots$. That is, $\tau_n(x_1, \ldots, x_n, x_{n+1}, \ldots) = (x_1, \ldots, x_n, a_{n+1}, \ldots)$ for all $p$ in $Y$. Now $\tau_n$ is continuous and hence $\mu_n = \mu^\tau_n$ is a finite signed measure on $Y$ concentrated in $Y_n$.

**Lemma 2.** If $f$ is in $C(Y)$, then

$$
\lim_{n \to \infty} \int_Y f d\mu_n = \int_Y f d\mu.
$$

**Proof.** First observe that

$$
\int_Y f(\phi) d\mu_n = \int_Y f(\phi) d\mu^\tau_n = \int_Y f(\tau_n(\phi)) d\mu \quad \text{for } n = 1, 2, \ldots.
$$

Then since $f \in C(Y)$ and $\tau_n$ converges uniformly to the identity map, we have

$$
\lim_{n \to \infty} \int_Y f(\phi) d\mu_n = \int_Y f(\phi) d\mu.
$$

Since $\mu_n$ is a finite signed measure concentrated in $Y_n$ it follows from [5, p. 288] that there exists a unique function $H_n$ on $Y_n$ such that

1. $H_n$ is bounded and of bounded variation on all $I \in S_n$,
2. $H_n(p) = 0$ if any $x_k = b_k$ when $p = (x_1, \ldots, x_n, a_{n+1}, \ldots)$,
3. $H_n$ is left continuous on $Y_n$ except possibly for points $p$.
On the other hand, the existence of an $H_n$ satisfying (1), (2), and (3) implies the existence of a measure $\mu_n$ concentrated on the Borel subsets of $Y_n$ such that (4) holds. We extend $H_n$ to be zero on $Y - Y_n$. Since $H_n = 0$ on $Y - Y_n$ it follows that

$$\int_Y (-1)^n H_n df = \sum_{I \in B_n} \int_I (-1)^n H_n df = \sum_{I \in B_n} \int_I (-1)^n H_n df.$$

Now by the integration by parts formula given in [6, p. 415] and the fact that $H^n(p) = 0$ for all $p = (x_1, \ldots, x_n, a_{n+1}, \ldots)$ when some $x_k = b_k$ for $k = 1, \ldots, n$, we obtain $\int_Y (-1)^n H_n df = \int_{Y_n} f dH_n$. It is now possible to relate the Riesz representation of a bounded linear functional and the representation provided in (3.1). In the next theorem we assume $\mu_n = \mu^n$ and that $H_n(p)$ is related to $\mu_n$ as above.

**Theorem 2.** If $F$ is a bounded linear functional on $C(Y)$ such that $F(f) = \int_Y f d\mu$ where $\mu$ is a finite signed measure on $\mathcal{B}$, then $F(f) = \int_Y h(df)$ for almost all $f$ in $C(Y)$ where $h \in \mathcal{L}_2(Y)$ and $h(p) = \lim_n (-1)^n H_n(p)$.

**Proof.** Since $\lim_n \int_{Y_n} f d\mu_n = \int_Y f d\mu$ and $\int_Y f d\mu_n = \int_Y f dH_n = \int_Y (-1)^n H_n df$ we have $F(f) = \lim_n \int_Y (-1)^n H_n df$ for all $f$ in $C(Y)$. Now $F(f) = \int_Y h(df)$ and since $\int_Y (-1)^n H_n df = \int_Y (-1)^n H_n(df)^n$ almost surely on $C(Y)$ it follows that $\lim_{n \to \infty} \int_Y [( -1)^n H_n - h](df)^n = 0$ for almost all $f$. However, $\int_Y [( -1)^n H_n - h](df)^n$ is a Gaussian functional with mean zero and variance $\int_Y [( -1)^n H_n - h]^2 dv$ so we have $\lim_n \int_Y [( -1)^n H_n - h]^2 dv = 0$ as was to be proved.

**Lemma 3.** If $E$ is an open subset of $C(Y)$, then $m(E) > 0$.

**Proof.** Let $I = \{f \in C(Y) : ||f|| \leq \lambda \}$ where $||f||$ is the uniform norm of $f$ and $\lambda > 0$. Let $\{f_k\}$ be a sequence of polynomials on $Y$ each in a finite number of variables such that $\{f_k\}$ is dense in $C(Y)$. Let $I_k = \{f \in C(Y) : ||f - f_k|| \leq \lambda \}$. Then $I_k - f_k = I$, and if $F(f) = \mathcal{L}_I(f)$, we have by the translation theorem in [3] that

$$E(F(f)) = E \left\{ F(f + f_k) \exp \left[ - \int_Y f_k^2 dv - 2 \int_Y f_k df \right] \right\}.$$

Thus $m(I) = 0$ if and only if $m(I_k) = 0$. However, $C(Y) = \bigcup_{k=1}^\infty I_k$ and if $m(I) = 0$ then $m(C(Y)) = 0$ which is a contradiction. Thus $m(I) > 0$ and $m(I_k) > 0$ for $k = 1, 2, \ldots$. Since $E$ is open there exists $\lambda > 0$ and $f_j$ such that $I_j = \{f \in C(Y) : ||f - f_j|| \leq \lambda \}$ is a subset of $E$. Hence $m(E) \geq m(I_j) > 0$ as was to be proved.
Let \( h(p) \) be a function on \( Y \) and define \( H_n(p) = h(p) \) for \( p \in Y \) except when some coordinate \( x_k \) of \( p \) is \( b_k \) and zero otherwise. Further, suppose \( H_n \) satisfies conditions (1) and (3) and that \( \mu_n \) is the finite signed measure concentrated on \( Y_n \) related to \((-1)^nH_n\). We then say \( \{\mu_n\} \) is obtained from \( h \). If \( \mu \) is a signed measure on \( Y \) then \( \|\mu\|_Y \) denotes the total variation of \( \mu \).

**Theorem 3.** If \( F(f) = \int_Y h(df) \) where \( h \in \mathcal{L}_2(Y) \) and \( \{\mu_n\} \) is obtained from \( h \) such that \( \|\mu_n\|_Y < M \) for \( n = 1, 2, \ldots \), then \( F(f) \) is essentially a bounded linear functional on \( C(Y) \), and for almost all \( f \) in \( C(Y) \)

\[
F(f) = \lim_{n \to \infty} \int_Y f \, d\mu_n.
\]

**Proof.** Since \( \int_Y fd\mu_n = \int_Y (-1)^n dH_n = \int_Y H_n df \) it follows that \( \lim_n \int_Y fd\mu_n = \lim_n \int_Y H_n df \). Now \( H_n \) converges to \( h \) in \( \mathcal{L}_2(Y) \), and for almost all \( f \in C(Y) \)

\[
\lim_{n \to \infty} \left[ \int_Y H_n df - \int_Y h(df) \right] = \lim_{n \to \infty} \int_Y (H_n - h)(df) = 0
\]

since \( \lim_n \int_Y (H_n - h)^2 dv = 0 \). Hence \( \lim_n \int_Y fd\mu_n = \int_Y h(df) = F(f) \) for almost all \( f \) in \( C(Y) \). Since open sets have positive measure there exists a dense set \( \{f_k\} \) in \( C(Y) \) such that \( \lim_n \int_Y f_k d\mu_n = F(f_k) \) for \( k = 1, 2, \ldots \). Let \( f \in C(Y) \). Then

\[
\left| \int_Y f \, d\mu_n - \int_Y f \, d\mu_m \right| \leq 2M\|f - f_k\| + \left| \int_Y f_k d\mu_n - \int_Y f_k d\mu_m \right|,
\]

so \( \lim_n \int_Y fd\mu_n \) exists for all \( f \in C(Y) \). If \( L(f) = \lim_n \int_Y fd\mu_n \), then \( L(f) \) is linear, \( L(f) = F(f) \) almost surely, and \( L(f) \leq M\|f\| \). Hence \( F(f) \) is an essentially bounded linear functional on \( C(Y) \).

4. Since any compact metric space \( X \) is homeomorphic to a closed subset of \( Y \) it follows that any additive functional on \( C(X) \) gives rise to an additive functional on \( C(Y) \). That is, let \( \phi \) be a homeomorphism of \( X \) into \( Y \) and define \( \theta(f) = f(\phi(\cdot)), f \in C(Y) \), then \( \theta \) maps \( C(Y) \) onto \( C(X) \) and \( \theta \) is continuous. The measure \( m^\theta \) is defined on the Borel subsets \( \mathcal{A} \) of \( C(Y) \) by the equation \( m^\theta(A) = m(\theta^{-1}(A)), A \in \mathcal{A} \). Then if \( G \) is an essentially additive \( m^\theta \)-measurable functional on \( C(X) \), we have \( F(f) = G(\theta(f)) \) as an essentially additive measurable functional on \( C(Y) \). Using Theorem 1 with \( \phi \) and \( \theta \) defined as above, we get the following result.

**Theorem 4.** If a functional \( G \) on \( C(X) \) is essentially additive and \( m^\theta \)-measurable, then there exists an \( h \) in \( \mathcal{L}_2(Y) \) such that \( g = \theta(f) \) implies
\[ G(g) = \int_Y h(d\mu) \]

for almost every \( g \) in \( C(X) \).

In particular, since the open sets of \( C(X) \) are \( m^g \)-measurable it follows that the bounded linear functionals on \( C(X) \) have the above representation.

**Bibliography**