NOTES ON COVERING TRANSFORMATION GROUPS

S. KINOSHITA[1]

1. Introduction. Let K and $\tilde{K}$ be topological spaces. A continuous map p of $\tilde{K}$ onto K is called a *covering map* if for any point a of K there exists an open neighborhood W of a such that $p^{-1}(W)$ is a union of disjoint open sets W_α and each $p|W_\alpha$ is a homeomorphism of W_α onto W. $\tilde{K}$ is called the *covering space* of K of p. A *covering transformation* is an autohomeomorphism g of $\tilde{K}$ such that $pg(x)=p(x)$ for every x of $\tilde{K}$. The *covering transformation group* of p is the set of all covering transformations with multiplication, which is the composite of transformations. A transformation group G of a topological space X is said to be *properly discontinuous* if for any $x\in X$ there is a neighborhood U of x such that for any $g\in G$, $g(U)\cap U=\emptyset$, where $g\neq e$. (Hence for any distinct g_1 and g_2 of G, $g_1(U)\cap g_2(U)=\emptyset$.) Let X be a connected topological space. Then a transformation group G of X is a covering transformation group of the projection map $p: X\to X/G$ if and only if G is properly discontinuous, where X/G is a topological space (see [**1**]). In this meaning, proper discontinuity characterizes the covering transformation group, but generally X/G may not be a Hausdorff space, while X is a manifold (see §4).

The purpose of this note is to introduce a condition, the *Sperner's condition*,[2] about G such that whenever X is a manifold, X/G is also a manifold. We may say that a transformation group G of a topological space X is said to satisfy the *Sperner's condition* if for any compact subset C of X the set

$$G[C] = \{g\in G \mid g(C)\cap C\neq\emptyset\}$$

is finite. A transformation group G of a topological space X is called fixed point free if any g of G $(g\neq e)$ has no fixed point. Then the main theorem of the note is as follows.

THEOREM. *The covering transformation group G of a covering map p of a connected topological space $\tilde{K}$ onto a locally compact Hausdorff space K is fixed point free and satisfies the Sperner's condition. Conversely, if G is a fixed point free transformation group of a connected locally com-*

Received by the editors January 30, 1967.

[1] Supported partly by NSF grant GP-5458.

[2] The condition introduced in the paper is slightly different from the original condition defined by Sperner [**2**]. A much more similar condition is defined in [**3**]. See also [**4**].

pact Hausdorff space X satisfying the Sperner's condition, then the projection map $p: X \to X/G$ is a covering map and X/G is a locally compact Hausdorff space. Further G is the covering transformation group of p.

Also a similar statement is true for a connected manifold, where a manifold is a separable, locally Euclidean Hausdorff space.

2. Let p be a covering map of a topological space $\tilde{K}$ onto a topological space K. It is clear that if K is a Hausdorff space, then so is $\tilde{K}$. If K is locally compact, so is $\tilde{K}$ and if K is regular, so is $\tilde{K}$.

If $\tilde{K}$ is connected, then any covering transformation group G is properly discontinuous. Therefore G is fixed point free.

To complete the proof of one direction of the main theorem, let us begin with a lemma. Let K be a Hausdorff space and p a covering map of $\tilde{K}$ onto K. Let a be a point of K and W a neighborhood of a defined in §1. Assume that U is a neighborhood of a such that $\overline{U} \subset W$. For each W_α choose a point $b_\alpha \in W_\alpha \cap p^{-1}(U)$. Put $B = \{b_\alpha\}$.

LEMMA. *B has no limit point, i.e., there is no point b of $\tilde{K}$ such that for any neighborhood V of b, $(V-b) \cap B \neq \emptyset$.*

PROOF. Suppose on the contrary that B has a limit point b. First we prove that $p(b) \in \overline{U}$. Let V be a neighborhood of $p(b)$. Since p is continuous, there is a neighborhood V_1 of b such that $p(V_1) \subset V$. Let $b_\alpha \in (V_1 - b) \cap B$. Then $p(b_\alpha) \in U$, and therefore $V \cap U \neq \emptyset$. Hence $p(b) \in \overline{U}$.

Now b is contained in some W_α. But since only one b_α is contained in W_α, b is not a limit point of B.

LEMMA. *The covering transformation group G of p, where p is a covering map of a connected topological space $\tilde{K}$ onto a regular space K, satisfies the Sperner's condition.*

PROOF. Suppose on the contrary that there is a compact subset C of $\tilde{K}$ such that $G[C]$ is not finite. For each $g_\beta \in G[C]$ choose a point $b_\beta \in g_\beta(C) \cap C$. Put $a_\beta = g_\beta^{-1}(b_\beta)$. Then $a_\beta \in C$. Hence there is a point a of C such that for any neighborhood V of a the set $\{g_\beta \in G[C] \mid a_\beta \in V\}$ is not finite. Let W be a neighborhood of $p(a)$ defined as in §1 and U a neighborhood of $p(a)$ such that $\overline{U} \subset W$. Put $U_\alpha = p^{-1}(U) \cap W_\alpha$ for each α. Then $U_\alpha \cap U_{\alpha'} = \emptyset$ for any $\alpha \neq \alpha'$. Let $a \in W_\gamma$. For each $g_\beta \in G[C]$ with $a_\beta \in U_\gamma$, b_β is contained in a U_β. Obviously the correspondence of such g_β to U_β is one to one. Hence the set $\{b_\beta \mid g_\beta \in [C],\ a_\beta \in U_\gamma\}$ has no limit point by the above lemma. On the other hand, we have assumed that $b_\beta \in C$ and $\{g_\beta \mid b_\beta \in G[C],\ a_\beta \in U_\gamma\}$ is not finite, which is a contradication.

We may note that any locally compact Hausdorff space is regular. Therefore, the proof of one direction of the main theorem is completed.

3. LEMMA. *Let G be a fixed point free transformation group of a locally compact Hausdorff space X. If G satisfies the Sperner's condition, then G is properly discontinuous.*

PROOF. Let U_1 be a neighborhood of $x \in X$ such that $\overline{U}_1$ is compact. Since $G[\overline{U}_1]$ is finite, there are only a finite number of $g(x)$, $g \in G$, such that $g(x) \in U_1$. Since G is fixed point free, there is a neighborhood U of x such that $\overline{U}$ does not contain any $g(x)$ $(g \neq e)$ and $U \subset U_1$. $G[\overline{U}]$ is also finite. Put $D = \bigcup_{g \in G[\overline{U}]-\{e\}} g(\overline{U})$. D is a finite sum of compact sets and hence compact. D does not contain the point x. Then there is a neighborhood V of x such that $V \subset U$ and $V \cap D = \emptyset$. Now it is easy to see that $g(V) \cap V = \emptyset$ for any $g \neq e$.

Let G be a fixed point free transformation group of a locally compact Hausdorff space X, satisfying the Sperner's condition. Then by the above arguments, the projection p of X onto the orbit space X/G is a covering map. Further if X is connected, then G is the covering transformation group of p.

LEMMA. *X/G is locally compact and Hausdorff.*

PROOF. It is clear that X/G is locally compact. We have to prove that if $x \neq g(y)$ for any $g \in G$, then there are neighborhoods U and V of x and y, respectively, such that $U \cap g(V) = \emptyset$ for any $g \in G$; that is to say, $g_1(U) \cap g_2(V) = \emptyset$ for any $g_1, g_2 \in G$.

Since $x \neq y$, there are neighborhoods U_1 and V_1 of x and y such that $U_1 \cap V_1 = \emptyset$ and that both $\overline{U}_1$ and $\overline{V}_1$ are compact. Put $C = \overline{V}_1 \cup \{x\}$. Since G satisfies the Sperner's condition, there are only a finite number of elements g of G such that $g(x) \in \overline{V}_1$. Then there exists a neighborhood V of y such that $\overline{V} \cap g(x) = \emptyset$ for any $g \in G$ and $V \subset V_1$. Now apply the Sperner's condition for $C' = \overline{U}_1 \cup \overline{V}$. Then only a finite number of images of $\overline{V}$ intersects with $\overline{U}_1$, and these images $g(\overline{V})$ do not contain the point x. Therefore there exists a neighborhood U of x such that $\overline{U} \cap g(\overline{V}) = \emptyset$ for any $g \in G$. Hence the proof is completed.

Now, by the above two lemmas, we complete the proof of the other direction of the main theorem.

4. REMARK. Let h be an orientation preserving fixed point free autohomeomorphism of a plane R^2. Then h induces a topological transformation group G, which is infinite cyclic. Such a G is always a properly discontinuous transformation group (see for instance [2]) and hence the projection $R^2 \to R^2/G$ is a covering map. But R^2/G is a

Hausdorff space if and only if G satisfies the Sperner's condition, hence if and only if h is topologically equivalent to a translation (see [2])[3]. This may justify consideration of a transformation group, which is not only properly discontinuous, but also to satisfy the Sperner's condition.

References

1. E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

2. E. Sperner, *Ueber die fixpunktfreien Abbildungen der Ebene*, Abh. Math. Sem. Univ. Hamburg **10** (1934), 1–47.

3. S. Kinoshita, *On quasi-translations in 3-space*, Fund. Math. **56** (1964), 69–79.

4. D. Montgomery and L. Zippin, *Translation groups in three-space*, Amer. J. Math. **49** (1937), 121–128.

5. P. S. Mostert, *One-parameter transformation groups in the plane*, Proc. Amer. Math. Soc. **9** (1958), 462–463.

Florida State University

[3] See also [4] and [5].