NOTE ON COLLAPSING $K \times I$ WHERE $K$ IS A CONTRACTIBLE POLYHEDRON

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The dunce hat $D$ is obtained from the two-simplex $\langle a, b, c \rangle$ by identifying all three sides, $\langle a, b \rangle = \langle a, c \rangle = \langle b, c \rangle$. $D$ is of interest because it is one of the simplest contractible polyhedra which is not collapsible (there is no free face from which to begin the collapsing). However, it is well known [2] that $D \times I$ is collapsible. This leads to the following conjecture.

Conjecture. If $K$ is a contractible two-complex, then $K \times I$ is collapsible. This conjecture is of particular interest since it implies the 3-dimensional Poincaré conjecture [2].

In this note we will consider a method for collapsing $K \times I$ for certain contractible polyhedra $K$. This method is summarized in the following theorem.

Theorem. If $L$ is a collapsible polyhedron and $L$ collapses to $K$ by an elementary collapse, then $K \times I$ is collapsible.

Proof. Since $Z$ collapses to $K$ by an elementary collapse, we have $L = K \cup B^n$ and $B^n \cap K = B^{n-1}$ with $B^{n-1} \subseteq \text{bdry}(B^n)$. $B^n$ and $B^{n-1}$ are polyhedral $n$ and $n-1$ balls respectively. Then $K \times I$ collapses to

$$(K \times \{0\}) \cup (B^{n-1} \times I) = K'.$$

$K'$ is clearly piecewise linearly homeomorphic to $L$. Thus $K \times I$ is collapsible.

As a trivial corollary of the previous theorem we get

Corollary. If $L$ is collapsible and $L$ collapses to $K$, then there is an integer $p$ such that $K \times I^p$ is collapsible.

Since by [1] a homotopically trivial polyhedron has the same simple homotopy type as a point, we have immediately

Corollary. If $K$ is a homotopically trivial polyhedron, then there is an integer $p$ such that $K \times I^p$ is collapsible.

Example 1. The dunce hat, $D$. Although it is well known that $D \times I$ is collapsible, an application of the above theorem seems to be conceptually simpler than the usual method.

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In Figure 1 we picture a two simplex, two of whose sides have been identified. The identification of a generator of the cone with its base, as indicated by the numbering of the vertices, yields the dunce hat.

We now expand $D$ to the complex $L = D \cup B^3$ where $B^3$ is the tetrahedron with vertices $v_0, v_1, v_3, v_4$. $L$ is indicated in Figure 2.

Now we note that $L$ collapses to $D$ (across $(v_1, v_3, v_4)$). Moreover it is easily seen that $L$ is collapsible. First collapse $B^3$ across $(v_0, v_1, v_3)$ and then proceed to collapse the two cell $(v_0, v_4) \cup (v_0, v_3, v_4)$ across the one cell $(v_0, v_3)$. The remaining collapses are obvious. Thus $D \times I$ is collapsible.

**Example 2.** Bing's house with two rooms, $H$, $H$ is the two-polyhedron pictured in Figure 3.
In Figure 3 we see that $T$ is a square disk with an open square disk removed, $P$ a square disk with two open square disks removed, and $B$ is a square disk with an open square disk removed. $W_1$, $W_2$, $W_3$, and $W_4$, the walls of the house, are square disks. $C_1$ and $C_2$, the two chimneys, are square cylinders, and $K_1$ and $K_2$, the curtains, are rectangular disks.

It is easy to show that $H$ is homotopically trivial, and clearly $H$ is not collapsible.
However, an application of the above theorem shows that $H \times I$ is collapsible. To see this just "fatten" the curtain $K_1$ up to a 3-cell $B^3$ as shown in Figure 4.

Let $K = H \cup B^3$. Clearly $K$ collapses to $H$ by an elementary collapse. Moreover, the following steps show that $K$ is collapsible.

1. Collapse $B^3$ across $B^3 \cap T$.
2. Collapse $B^3 \cap C_1$ across $B^3 \cap C_1 \cap T$.
3. Collapse $B^3 \cap W_4$ across $B^3 \cap W_4 \cap T$.
4. Collapse $B^3 \cap P$ across $B^3 \cap P \cap C_1$.

We have now reached a position where we may collapse across the one cell $B^3 \cap W_4 \cap P$. After several collapses one can eliminate the entire bottom room along with its chimney and curtain. The remaining collapses should be clear.

References


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