GROUPS IN WHICH EVERY MAXIMAL PARTIAL ORDER IS ISOLATED

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A partial order on a group G is said to be isolated iff for all x in G and all $n>0$, $e \leqq x^n$ implies $e \leqq x$. (e denotes the identity element of G.) Fuchs [1] posed the problem of determining necessary and sufficient conditions for a group to have the property that every partial order can be extended to an isolated partial order. We will call such groups *I^* groups*. Theorem 2 will provide such conditions. In the balance of the paper we examine some of the properties of I^* groups and their relation to O^* groups.

Following Fuchs [1] ,we will denote the normal subsemigroup of G generated by x by $S(x)$. $S(x, y)$ is defined similarly. x is said to be *generalized periodic* if $e \in S(x)$ and *nongeneralized periodic* otherwise. We let $S'(x) = S(x) \cup \{e\}$.

We will be concerned with the following group-theoretic condition:

CONDITION A. For all $x \in G$, if $t \in S(x)$ and $n>0$, then $e \in S(t^{-1}, x^n)$.

LEMMA 1. *If the group G satisfies condition A and $e \in S(x)$, then $e \in S(x^n)$ for all $n>0$.*

PROOF. If $e \in S(x)$, then $x^{-1} \in S(x)$ so $x^{-n} \in S(x)$. Condition A implies $e \in S((x^{-n})^{-1}, x^n) = S(x^n)$.

THEOREM 2. *G is an I^* group iff G is torsion free and G satisfies Condition* A.

PROOF. If G is an I^* group, then clearly G is torsion free. Let $t \in S(x)$ and $n>0$. If $e \notin S(t^{-1}, x^n)$, then $S'(t^{-1}, x^n)$ is the cone of a partial order which cannot be extended to an isolated partial order.

Now let G be torsion free and assume that Condition A is satisfied. Let P be the cone of a maximal partial order. If $P = \{e\}$, then P is isolated since G is torsion free. Let $P \neq \{e\}$ and suppose $x^n \in P$ and $x \neq e$. Either $x \in P$ or $S(x) \cap P^{-1} \neq \emptyset$. (If $S(x) \cap P^{-1} = \emptyset$, $S'(x)P$ is a cone containing P and the maximality of P implies then that $S'(x)P = P$.) Suppose $S(x) \cap P^{-1} \neq \emptyset$ and let $t \in S(x) \cap P^{-1}$. $x^n \in P$ implies $e \notin S(x^n)$ so $e \notin S(x)$ by Lemma 1. Thus $t \neq e$ and therefore $S(t)$

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$\subseteq (P^{-1}-\{e\})$ and $S(x^n)\subseteq (P-\{e\})$. Since $e\notin S(t)$ and $e\notin S(x^n)$, $e\in S(t^{-1}, x^n)$ implies $S(t)\cap S(x^n)\neq\emptyset$. From this it follows that $(P^{-1}-\{e\})\cap(P-\{e\})\neq\emptyset$ so that $P^{-1}\cap P\neq\{e\}$ which is impossible. Therefore $S(x)\cap P^{-1}=\emptyset$ and $x\in P$.

As one would expect, a *locally I^* group* is a group in which every finitely generated subgroup is I^*. Since Condition A and the property of being torsion free are both local properties we have:

COROLLARY 3. *If G is locally I^*, then G is an I^* group.*

PROPOSITION 4. *If G satisfies Condition* A *and H is a homomorphic image of G, then H also satisfies Condition* A.

PROOF. Let f be a homomorphism from G onto H, let $h\in H$, $t\in S(h)$, and $n>0$. For some $x\in G, f(x)=h$ and for some $r\in S(x), f(r)=t$. Since $e\in S(r^{-1}, x^n)$, $e\in f(S(r^{-1}, x^n))\subseteq S(f(r^{-1}), f(x^n))=S(t^{-1}, h^n)$.

COROLLARY 5. *A torsion free image of a group satisfying Condition A is an I^* group. In particular a torsion free image of an I^* group is an I^* group.*

Ohnishi [**4**] introduced the following group theoretic condition:

CONDITION B. For all x in G, if $t\in S(x)$ and $u\in S(x)$, then $S(t)\cap S(u)\neq\emptyset$.

He showed that Condition B and nongeneralized periodicity are necessary and sufficient conditions for a group to be an O^* group. (An O^* group is a group in which every partial order can be extended to a full order.)

PROPOSITION 6. *Condition* B *implies Condition* A.

PROOF. Let $t\in S(x)$ and $n>0$. Since $x^n\in S(x)$, Condition B implies $S(t)\cap S(x^n)\neq\emptyset$ so $e\in S(t^{-1}, x^n)$.

Corollary 5 and Proposition 6 together imply:

COROLLARY 7. *A torsion free image of an O^* group is an I^* group.*

Wiegold [**5**] showed that the group $K=\langle x, y: x^3=y^2, xy(x^2y)^5xy^2=e\rangle$ is torsion free. If we let $Z(K)$ denote the center of K and let P be a maximal partial order of K, then $P\cap Z(K)$ is a full order on $Z(K)$. Since $x^3=y^2\in Z(K)$, $x^3=y^2\in P\cup P^{-1}$ and without loss of generality we may assume $x^3=y^2\in P$. Because of the second relation, it is impossible for both x and y to be elements of P. Therefore P is not isolated and K is not an I^* group. Since K is an image of every nonabelian free group, we have shown that the nonabelian free groups are not I^* groups. Thus the nonabelian free groups provide examples of O groups which are not I^* groups. Since all O^* groups are I^* groups, the above argument provides another proof that nonabelian free groups are not O^* groups [2].

Kurosh [3, Vol. 2, p. 157] has shown that every torsion free group can be embedded in a torsion free group with only two conjugate classes. If H is a torsion free group, denote by H^θ the group obtained in this manner. If we consider the group $Z \oplus H^\theta$, where Z denotes the integers, it can be shown that any maximal cone is of the form

$$P = \{(m, h) : m = 0 \quad \text{and} \quad h = e \text{ or } m > 0\}$$

or the dual of such a cone. P is easily seen to be isolated and directed. Thus any torsion free group can be embedded in an I^* group in which every maximal partial order is directed. Note, however, that in the above construction the partial order induced on the subgroup H is always trivial. Notice also that not every subgroup of an I^* group need be an I^* group, *e.g.* $Z \oplus K^\theta$ is I^* but K is not.

References

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