Heywood [4] and others have considered integrability theorems for power series and Laplace transforms that are analogous to known results for Fourier series and transforms. These results are all weighted $L^1$ results. Here we obtain an $L^p$ theorem which is analogous to the well-known $L^p$ result of Hardy and Littlewood [3].

**Theorem.** Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k \geq 0$, $0 \leq x < 1$. Then for $1 \leq p \leq \infty$,

\[
\left[ \int_0^1 |f(x)|^p \, dx \right]^{1/p} < \infty
\]

if and only if

\[
\left( \sum_{n=1}^{\infty} n^{-2} \left( \sum_{k=0}^{n} a_k \right)^p \right)^{1/p} < \infty.
\]

In the analogous theorem for Fourier series the added condition $a_{n+1} \leq a_n$, $a_n \to 0$, was assumed by Hardy and Littlewood and they stated condition (2) as $\sum n^{p-2} a_n^p < \infty$. Boas [1] has some partial results for Fourier series on $L^p$ theorems with only the condition $a_n \geq 0$ but only for some $L^p$ spaces with a singular weight function and the question for $L^p$ without a weight seems difficult. For further information see [2].

We first show that if $a_n \geq 0$, then (1) implies (2). We may assume $1 < p < \infty$, as the two end cases are trivial. Then

\[
\int_0^1 [f(x)]^p \, dx = \sum_{n=1}^{\infty} \int_{1-1/n}^{1-1/(n+1)} [f(x)]^p \, dx
\]

\[
= \sum_{n=1}^{\infty} \int_{1-1/n}^{1-1/(n+1)} \left[ \sum_{k=0}^{n} a_k x^k \right]^p \, dx
\]

\[
= \sum_{n=1}^{\infty} \int_{1-1/n}^{1-1/(n+1)} \left[ \sum_{k=0}^{n} a_k \left( 1 - \frac{1}{n+1} \right)^k \right]^p \, dx
\]

\[
= A \sum_{n=1}^{\infty} n^{-2} \left[ \sum_{k=0}^{n} a_k \right]^p
\]

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since \((1 - 1/(n + 1))^k \geq A > 0, k = 0, 1, \ldots, n\).

In the other direction we have

\[
\int_0^1 [f(x)]^p dx = \sum_{n=2}^{\infty} \int_{1-1/(n-1)}^{1} [f(x)]^p dx 
\leq A \sum_{n=2}^{\infty} n^{-2} \left[ \sum_{k=0}^{\infty} a_k \left(1 - \frac{1}{n}\right)^k \right]^p
\leq A \sum_{n=2}^{\infty} n^{-2} \left[ \sum_{k=0}^{\infty} \sum_{j=nk} a_j \left(1 - \frac{1}{n}\right)^j \right]^p
\leq A \sum_{n=2}^{\infty} n^{-2} \left( \sum_{k=0}^{\infty} e^{-k} \sum_{j=nk} a_j \right)^p
\leq A \sum_{n=2}^{\infty} n^{-2} \left( \sum_{k=0}^{\infty} e^{-(k/2)^p/p'} \left( \sum_{j=0}^{\infty} e^{-(kp/2)} \left[ \sum_{j=0}^{\infty} a_j \right]^p \right) \right)
\leq A \sum_{n=2}^{\infty} n^{-2} \sum_{k=0}^{\infty} e^{-(kp/2)} \left[ \sum_{j=0}^{\infty} a_j \right]^p
= A \sum_{k=0}^{\infty} e^{-(kp/2)} \sum_{n=2}^{\infty} n^{-2} \left[ \sum_{j=0}^{\infty} a_j \right]^p
\leq A \sum_{k=0}^{\infty} (k + 1)^2 e^{-(kp/2)} \sum_{n=2}^{\infty} \left[ (k + 1)n \right]^{-2} \left[ \sum_{j=0}^{\infty} a_j \right]^p
\leq A \sum_{n=2}^{\infty} n^{-2} \left[ \sum_{j=0}^{n} a_j \right]^p.
\]

This result can undoubtedly be extended in some of the many ways that Heywood’s original \(L^1\) result was extended. We leave these extensions to the interested reader.

Using a result of Konyushkov [5] we have the following corollary.

**Corollary.** Let \(f(x) = \sum a_n x^n, a_{n+1} \leq a_n\). Then for \(1 < p < \infty\) we have \(f(x) \in L^p(0, 1)\) if and only if \(f(e^{i\theta}) \in L^p(-\pi, \pi)\).

This follows since \(f(e^{i\theta}) \in L^p\) if and only if \(\sum a_n^p n^{p-2} < \infty\) by the Hardy-Littlewood Theorem and by Konyushkov’s result, \(\sum a_n^p n^{p-2} < \infty\) if and only if \(\sum n^{-2} [\sum a_k]^p < \infty\). Each of these results also holds for quasi-monotone sequences and thus so does the corollary. That \(f(e^{i\theta}) \in L^p(-\pi, \pi)\) implies \(f(x) \in L^p(0, 1)\) is a classical
result and requires no condition on the coefficients. It does not seem obvious that this is reversible with monotone coefficients.

References