Lp BEHAVIOR OF POWER SERIES WITH
POSITIVE COEFFICIENTS

RICHARD ASKEY

Heywood [4] and others have considered integrability theorems
for power series and Laplace transforms that are analogous to known
results for Fourier series and transforms. These results are all
weighted \( L^1 \) results. Here we obtain an \( L^p \) theorem which is analogous
to the well-known \( L^p \) result of Hardy and Littlewood [3].

**Theorem.** Let \( f(x) = \sum_{k=0}^{\infty} a_k x^k \), \( a_k \geq 0 \), \( 0 \leq x < 1 \). Then for \( 1 \leq p \leq \infty \),

\[
\left( \int_0^1 |f(x)|^p dx \right)^{1/p} < \infty
\]

if and only if

\[
\left( \sum_{n=1}^{\infty} n^{-2} \left( \sum_{k=0}^{n-1} a_k \right)^p \right)^{1/p} < \infty.
\]

In the analogous theorem for Fourier series the added condition
\( a_{n+1} \leq a_n \), \( a_n \to 0 \), was assumed by Hardy and Littlewood and they
stated condition (2) as \( \sum n^{p-2} a_n^p < \infty \). Boas [1] has some partial
results for Fourier series on \( L^p \) theorems with only the condition
\( a_n \geq 0 \) but only for some \( L^p \) spaces with a singular weight function and
the question for \( L^p \) without a weight seems difficult. For further
information see [2].

We first show that if \( a_n \geq 0 \), then (1) implies (2). We may assume
\( 1 < p < \infty \), as the two end cases are trivial. Then

\[
\int_0^1 [f(x)]^p dx = \sum_{n=1}^{\infty} \int_{1-1/n}^{1-1/(n+1)} [f(x)]^p dx
\]

\[
\geq \sum_{n=1}^{\infty} \int_{1-1/n}^{1-1/(n+1)} \left[ \sum_{k=0}^{n} a_k x^k \right]^p dx
\]

\[
\geq \sum_{n=1}^{\infty} \int_{1-1/n}^{1-1/(n+1)} \left[ \sum_{k=0}^{n} a_k \left( 1 - \frac{1}{n+1} \right)^k \right]^p dx
\]

\[
\geq A \sum_{n=1}^{\infty} n^{-2} \left[ \sum_{k=0}^{n} a_k \right]^p
\]

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since \((1 - 1/(n+1))^k \geq A > 0, k = 0, 1, \ldots, n\).

In the other direction we have

\[
\int_0^1 [f(x)]^p \, dx = \sum_{n=2}^{\infty} \int_{1-1/(n-1)}^{1-1/n} [f(x)]^p \, dx
\]

\[
\leq A \sum_{n=2}^{\infty} n^{-2} \left[ \sum_{k=0}^{\infty} a_k (1 - 1/n)^k \right]^p
\]

\[
\leq A \sum_{n=2}^{\infty} n^{-2} \left[ \sum_{k=0}^{\infty} \sum_{j=kn}^{n(k+1)} a_j \left(1 - \frac{1}{n}\right)^j \right]^p
\]

\[
\leq A \sum_{n=2}^{\infty} n^{-2} \left[ \sum_{k=0}^{\infty} \sum_{j=0}^{n(k+1)} a_j \right]^p
\]

\[
\leq A \sum_{n=2}^{\infty} n^{-2} \left( \sum_{k=0}^{\infty} e^{-(k/2)p} \right)^{p/p'} \left( \sum_{k=0}^{\infty} \sum_{j=0}^{k-n+1} a_j \right)^p
\]

\[
= A \sum_{k=0}^{\infty} e^{-(k/2)p} \sum_{n=2}^{\infty} n^{-2} \left[ \sum_{j=0}^{n(k+1)} a_j \right]^p
\]

\[
\leq A \sum_{k=0}^{\infty} (k+1)^2 e^{-(k/2)p} \sum_{n=2}^{\infty} [(k+1)n-2]^{-2} \left[ \sum_{j=0}^{n(k+1)} a_j \right]^p
\]

\[
\leq A \sum_{n=2}^{\infty} n^{-2} \left[ \sum_{j=0}^{n} a_j \right]^p.
\]

This result can undoubtedly be extended in some of the many ways that Heywood's original \(L^1\) result was extended. We leave these extensions to the interested reader.

Using a result of Konyushkov [5] we have the following corollary.

**Corollary.** Let \(f(x) = \sum a_n x^n, a_{n+1} \leq a_n\). Then for \(1 < p < \infty\) we have \(f(x) \in L^p(0, 1)\) if and only if \(f(e^{i\theta}) \in L^p(-\pi, \pi)\).

This follows since \(f(e^{i\theta}) \in L^p\) if and only if \(\sum a_n n^{-p-2} < \infty\) by the Hardy-Littlewood Theorem and by Konyushkov’s result, \(\sum a_n n^{-p-2} < \infty\) if and only if \(\sum n^{-2} [\sum a_k]^p < \infty\). Each of these results also holds for quasi-monotone sequences and thus so does the corollary. That \(f(e^{i\theta}) \in L^p(-\pi, \pi)\) implies \(f(x) \in L^p(0, 1)\) is a classical
result and requires no condition on the coefficients. It does not seem obvious that this is reversible with monotone coefficients.

References


University of Wisconsin