A SPACE FOR WHICH $H^1(X; Z) \not\approx [X, S^1]$[1]

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It is well known that $H^1(X; Z) \approx [X, S^1]$ for paracompact (Hausdorff) spaces X, where cohomology is the sheaf theoretic (Grothendieck) cohomology and $[X, S^1]$ is the group of homotopy classes of maps of X into the circle (see [1, p. 114]). This fact was generalized by Huber [3] to higher cohomology groups replacing S^1 by certain Eilenberg-Mac Lane spaces.

We shall show that this result does not generalize to arbitrary spaces. In fact we shall construct a locally compact Hausdorff space X for which $H^1(X; Z) \approx Z$ and $[X, S^1] = 0$. The question is still open for normal spaces X.

We begin by considering the "compactified long line $\tilde{L}$." This is a space with the following properties: (a) $\tilde{L}$ is compact Hausdorff, connected and simply ordered with the order topology. There is a maximal element $x \in \tilde{L}$ (with respect to the given order) and $L = \tilde{L} - \{x\}$ is the "long ray." (b) For any continuous real valued function f on L or on $\tilde{L}$, f is constant on some interval $[y, x]$ for $y < x$ and the same property follows for any map into a separable metric space. (c) For any countable subset $K \subset L$, there exists an element $y \in L$ with $k < y$ for all $k \in K$.

These properties of L and $\tilde{L}$ are all that we shall use.

Now let $\tilde{L}_1$ and $\tilde{L}_2$ be copies of $\tilde{L}$ with x_1, x_2 corresponding to x. Let A be the one point union of $\tilde{L}_1$ and $\tilde{L}_2$ which identifies x_1 with x_2. (We shall denote the corresponding subsets of A by $\tilde{L}_i$ and the common point by x.) Let I be the real interval $[-1, 1]$ and define

$$\tilde{X} = A \times I, \qquad X = \tilde{X} - \{x \times 0\}.$$

We shall first prove that $H^1(X; Z) \approx Z$. Define $X_1 = (A \times [0, 1]) \cap X$ and $X_2 = (A \times [-1, 0]) \cap X$. Now X_1 has $A \times \{1\}$ as a strong deformation retract so that

$$H^*(X_1; Z) \approx H^*(A; Z)$$

by the homotopy property (see [1, Chapter II, §11.4]). But A is a simply ordered compact space with the order topology so that $H^n(A; Z) = 0$ for $n > 0$ by [1, Exercise 2, p. 108]. Since A is connected $H^0(A; Z) \approx Z$.

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Now $X_1 \cap X_2$ consists of two disjoint copies of L and by [1, Exercise 3, p. 108] we see that

$$\begin{aligned} H^n(L; Z) &\approx Z, \qquad n = 0, \\ &\approx 0, \qquad n \neq 0. \end{aligned}$$

The exact Mayer-Vietoris sequence for (X_1, X_2) (see [1, Chapter II, §13, (1), p. 65]) shows that

$$\begin{aligned} H^n(X; Z) &\approx Z, \qquad n = 0, 1, \\ &\approx 0, \qquad n \neq 0, 1. \end{aligned}$$

Now we claim that $[X, S^1] = 0$. Let $f: X \to S^1$ be continuous. Let $r \in [-1, 1]$ and consider the restriction of f to $A \times \{r\}$ (or $A \times \{0\} - \{x \times 0\}$ for $r = 0$). By property (b) there exists an element $y(r) \in L$ corresponding to $y_1(r)$ and $y_2(r)$ in L_1 and L_2 such that f is constant on the interval $[y_1(r), y_2(r)] \times \{r\}$. [Here we are ordering A by taking $\tilde{L}_1$ first and $\tilde{L}_2$ second, reversing the order on $\tilde{L}_2$.] By property (c) there exists $y \in L$ corresponding to $y_i \in L_i$ such that $y > y(r)$ for all *rational* $r \in I$. Thus $[y_1, y_2] \subset [y_1(r), y_2(r)]$ for r rational. Now f is constant on $[y_1, y_2] \times \{r\}$ for r rational and continuity shows that it is also constant for arbitrary r. Thus on $[y_1, y_2] \times I$, f is given by composing the projection onto I with a continuous function on I. It clearly follows that f extends to a continuous map

$$\tilde{f}: \tilde{X} \to S^1.$$

It suffices to show that $[\tilde{X}, S^1] = 0$.

Since $\tilde{X}$ is compact, we have $[\tilde{X}, S^1] \approx H^1(\tilde{X}; Z)$. But $\tilde{X}$ has the homotopy type of A and A is acyclic, as we have remarked above, so that $[\tilde{X}, S^1] \approx H^1(\tilde{X}; Z) \approx H^1(A; Z) = 0$ as was to be shown.

REMARK 1. It can be shown that X is a cohomology 2-manifold with boundary, but of course it is not locally euclidean.

REMARK 2. It may be of interest to give briefly another example which led up to the construction of the present one. It is not so nice a space but it is Hausdorff and satisfies the second axiom of countability. Moreover the proofs are much simpler for it. We start with a Hausdorff second axiom space Y which is connected but has only a countably infinite number of points. Such a space is constructed in [2]. Let $\{a, b\}$ be two points of Y and let Y' be obtained from two copies of Y by identifying the points corresponding to a and those corresponding to b. Then the Mayer-Vietoris sequence (cited above) shows that $H^1(Y'; Z) \neq 0$. However, let $f: Y' \to W$ be any map into a completely regular space W. Then $f(Y')$ is connected, completely

regular and countable. We claim $f(Y')$ consists of exactly one point, for if, on the contrary, $w_1 \neq w_2$ are in $f(Y')$, take a real valued continuous function g on W separating these points and note that $gf(Y')$ is connected, hence it is an interval, and is countable. In particular $[Y', S^1]=0$.

REMARK 3. Although we have used sheaf cohomology, it is also true that $\check{H}^1(X; Z) \neq 0$ for Čech theory. [It is known, in fact, that $\check{H}^1 \approx H^1$ for general spaces and arbitrary coefficient sheaves.] Moreover, $\check{H}^1(X; Z) \neq 0$ for Čech theory defined by means of *any* directed class of coverings containing the finite coverings and also for coefficients in the constant *sheaf* Z *or* in the constant *presheaf* Z. To see this one notes that X possesses a finite covering by connected open sets whose nerve is topologically a circle. Then one recalls that for *sheaf* coefficients the refinement homomorphisms on *one-dimensional* cohomology are always monomorphisms. This proves the assertion for *sheaf* coefficients and it follows for *presheaf* coefficients by using the fact that the given covering consists of connected sets and by using the natural homomorphism from the presheaf case to the sheaf case.

REMARK 4. Our example also provides an example of a sheaf, on a locally compact space, which is c-soft yet not acyclic. In fact let $\mathfrak{F}$ be the sheaf of germs of continuous real valued functions on X. Then, in general, there is an exact sequence (see [**1**, p. 114])

$$0 \to [X, S^1] \to H^1(X; Z) \to H^1(X; \mathfrak{F}).$$

Then $\mathfrak{F}$ is c-soft but, with X as above, $H^1(X; \mathfrak{F}) \neq 0$ so that $\mathfrak{F}$ is not acyclic.

REFERENCES

1. G. E. Bredon, *Sheaf theory*, McGraw-Hill, New York, 1967.

2. S. W. Golomb, *A connected topology for the integers*, Amer. Math. Monthly **66** (1959), 663–665.

3. P. J. Huber, *Homotopical cohomology and Čech cohomology*, Math. Ann. **144** (1961), 73–76.

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