1. The main results. Let \( \overline{R} \) be a compact bordered Riemann surface with interior \( R \). We represent \( R \) as the orbit space \( \Delta/G \) where \( G \) is a finitely generated Fuchsian group of the second kind acting on the unit disk \( \Delta \). Choose a fundamental polygon \( \mathfrak{a} \) for \( G \) in \( \Delta \) whose closure \( \overline{\mathfrak{a}} \) in the plane meets the boundary of \( \Delta \) in a finite number of arcs, each of which corresponds to a boundary contour of \( R \).

On the set of analytic functions in \( \Delta \) we will consider the norms

\[
(1) \quad \|f\|_\infty = \sup \{ |f(z)| : z \in \Delta \},
\]

\[
(2) \quad \|f\| = \iint_\Delta |f(z)| \, dxdy,
\]

and the corresponding Banach spaces

\[
H^\infty(\Delta) = \{ f : \|f\|_\infty < \infty \} \quad \text{and} \quad A(\Delta) = \{ f : \|f\| < \infty \}.
\]

We shall also consider the subspaces \( H^\infty(G) \subset H^\infty(\Delta) \) and \( A(G) \subset A(\Delta) \) of functions which satisfy

\[
(3) \quad f(Az) = f(z) \quad \text{for all } A \in G \quad \text{and} \quad z \in \Delta.
\]

If \( f \) satisfies (3), then

\[
(4) \quad \iint_\Delta |f(z)| \, dxdy = \iint_{\mathfrak{a}} \left| f(z) \right| \left( \sum_{A \in G} |A'(z)|^2 \right) \, dxdy
\]

so that \( A(G) \) consists of those analytic functions which satisfy (3) and are summable over \( \mathfrak{a} \) with respect to the measure

\[
(5) \quad dm(z) = \sum_{A \in G} |A'(z)|^2 \, dxdy.
\]

Thus \( H^\infty(G) \) corresponds to the space of bounded analytic functions on \( R \), and \( A(G) \) to the space of analytic functions on \( R \) which are summable with respect to \( dm \).

Theorem. There is a projection \( P \), bounded in the respective norms, which sends \( A(\Delta) \) (resp. \( H^\infty(\Delta) \)) onto \( A(G) \) (resp. \( H^\infty(G) \)) and has the following property: if \( f(\xi) = f(A\xi) \) for some \( \xi \in \Delta \) and all \( A \in G, f \in A(\Delta) \), then for any \( g \in A(\Delta) \)
(6) \((Pfg)(\xi) = f(\xi)Pg(\xi)\).

In particular, if \(f \in A(G)\),

(7) \(P(fg) = fPg\).

This Theorem is a direct consequence of the following.

**Lemma.** There is a polynomial \(F(z)\) such that the Poincaré series

(8) \(\Theta F(z) = \sum_{A \in G} F(Az)A'(z)^2\)

is bounded away from zero in the fundamental polygon \(R\).

A somewhat less general form of the Theorem is due to Forelli [6] who obtained a bounded projection \(P\) of \(H^\infty(\Delta)\) onto \(H^\infty(G)\) having property (7).

2. Proof of the Lemma. The set of limit points of \(G\) is a closed subset of the unit circle of linear measure zero. If \(\Omega\) is the complement of the set of limit points in the extended plane, then \(\Omega/G\) is a compact Riemann surface, the double of \(\mathbb{R}\). Let \(\mathbb{R}^* \supseteq \overline{\mathbb{R}}\) be a subsurface of \(\Omega/G\) such that (i) \(\mathbb{R}^*\) is bounded by analytic curves, (ii) each component of \(\mathbb{R}^* - \overline{\mathbb{R}}\) is a topological annulus, and (iii) \(\pi(\infty)\) is in the exterior of \(\mathbb{R}^*\) where \(\pi: \Omega \to \Omega/G\) is the natural map. Then \(\pi^{-1}(\mathbb{R}^*) = D^*\) contains \(\Delta\), is invariant under \(G\), and is bounded by a Jordan curve which is the union of \(\pi^{-1}(\partial \mathbb{R}^*)\) and the set of limit points of \(G\). Moreover, \(\mathbb{R}\) is a compact subset of \(D^*\). Let \(\mathbb{R}^*\) be a fundamental "polygon" for \(G\) in \(D^*\) (\(\mathbb{R}^*\) can be obtained, for example, by mapping \(D^*\) onto \(\Delta\)).

By Abel's theorem there exists a meromorphic differential \(\omega\) on the compact surface \(\Omega/G\) which is analytic and nonzero on the closure of \(\mathbb{R}^*\). The quadratic differential \(\omega^2\) can be lifted to \(D^*\) to determine an analytic \(\phi(z)\) which is nonzero in \(D^*\) and satisfies

(9) \(\phi(Az)A'(z)^2 = \phi(z)\) for all \(z \in D^*, A \in G\).

Furthermore, since \(\omega^2\) is analytic in the closure of \(\mathbb{R}^*\),

(10) \(\int \int_{\mathbb{R}^*} |\phi(z)| \, dx dy < \infty\).

We now appeal to a recent theorem of Bers [3] concerning Poincaré series in \(D^*\). Let \(Q(G)\) denote the Banach space of all functions \(\phi(z)\) analytic in \(D^*\) which satisfy (9) and (10), the norm being given by (10). Bers has proved [3, Theorem 2] that the Poincaré series (8) defines a continuous map of \(A(D^*)\) onto \(Q(G)\). (A short proof of this
theorem can be found in [4].) Furthermore, since $D^*$ is a Jordan region, a theorem of O. J. Farrell [5] implies that the polynomials are dense in $A(D^*)$.

Applying these results to our nonzero function $\phi(z)$ in $Q(G)$, we obtain a sequence $\{F_n\}$ of polynomials such that $\Theta F_n \rightarrow \phi$ in $Q(G)$. But convergence in $Q(G)$ implies uniform convergence on compact sets in $D^*$. In particular, $\Theta F_n \rightarrow \phi$ uniformly in $\mathfrak{R}$, and for sufficiently large $n$, $\Theta F_n$ is bounded away from zero in $\mathfrak{R}$. q.e.d.

3. Proof of the Theorem. Choose a polynomial $F(z)$ in accordance with the Lemma. For $f \in A(\Delta)$ define

$$P_f(z) = (\Theta F)(z)/\Theta F(z).$$

Set

$$\delta^{-1} = \inf \{ | \Theta F(z) | : z \in \mathfrak{R} \} > 0,$$

$$M = \sup \left\{ \sum_{A \in G} | A'(z) |^2 : z \in \mathfrak{R} \right\} < \infty.$$

Then

$$\int \int_{\mathfrak{R}} | \Theta Ff | \, dx \, dy \leq \| F \|_\infty \int \int_{\mathfrak{R}} \sum_{A \in G} | f(Az) | \cdot | A'(z) |^2 \, dx \, dy$$

$$= \| F \|_\infty \sum_{A \in G} \int \int_{A(\mathfrak{R})} | f(z) | \, dx \, dy = \| F \|_\infty | f |.$$  

Hence the series $\Theta Ff$ converges absolutely and uniformly on compact subsets of $\mathfrak{R}$ and therefore on compact subsets of $\Delta$. Furthermore by the Lemma, $\Theta F$ is nonzero on $\Delta$. Consequently, $P_f$ is analytic in $\Delta$; $P_f$ obviously satisfies (3) and (6). We also have from (4) and (12) that

$$\| P_f \| = \int \int_{\Delta} | P_f(z) | \, dx \, dy = \int \int_{\mathfrak{R}} | P_f(z) | \, dm(z)$$

$$\leq M \int \int_{\mathfrak{R}} | P_f(z) | \, dx \, dy \leq M \delta \int \int_{\mathfrak{R}} | \Theta Ff | \, dx \, dy$$

$$\leq M \delta \| F \|_\infty | f |,$$

and, if $f$ is in $H^\omega(\Delta)$ as well,

$$\| P_f \|_\infty = \sup \{ | P_f(z) | : z \in \Delta \}$$

$$= \sup \{ | P_f(z) | : z \in \mathfrak{R} \} \leq M \delta \| F \|_\infty | f |_\infty.$$

The remainder of the Theorem follows at once. q.e.d.
4. Applications. As Forelli has pointed out [6, Corollary 2], Carleson's solution of the corona problem for $H^\infty(\Delta)$ and the existence of a projection $P: H^\infty(\Delta) \to H^\infty(G)$ with property (7) yield a solution of the corona problem for the compact bordered surface $R$. We state this consequence of Theorem 1 as

**Corollary 1.** Let $\mathfrak{M}(R)$ be the maximal ideal space of the algebra $H^\infty(G)$. Then $R$ is dense in $\mathfrak{M}(R)$.

Other proofs of Corollary 1 have been given in [1], [6], [7].

**Corollary 2.** Let $S$ be a set of points in $\Delta$ which is invariant under $G$ and let $\xi(z)$ be a complex valued function on $S$ such that $\xi(Az) = \xi(z)$ for all $A \in G$ and $z \in S$. There exists $f \in A(G)$ (resp. $H^\infty(G)$) with $f(z) = \xi(z)$ for all $z \in S$ if and only if there exists $g \in A(\Delta)$ (resp. $H^\infty(\Delta)$) with $g(z) = \xi(z)$, all $z \in S$.

Indeed, if $g(z)$ is given, then $f(z) = Pg(z) = \xi(z)$ for all $z \in S$.

Corollary 2 strengthens Stout's theorem [7] that if $S$ is a $G$-invariant interpolating set for $H^\infty(\Delta)$ it is also an interpolating set for $H^\infty(G)$.

Let $A(G) \subseteq H^\infty(G)$ be the subspace of functions continuous in $\overline{G}$, and $A_0(G) \subseteq A(G)$ the subset of functions analytic in $\overline{G}$.

**Corollary 3.** $A_0(G)$ is dense in $A(G)$.

**Proof.** We use the notations of §3. If $f \in A_0(G)$, set $f_r(z) = f(rz)$ for $r < 1$. Then $P(f_r) \in A_0(G)$. We claim that $P(f_r) \to P(f) = f$ as $r \to 1$. For the proof, choose an enumeration $\{A_i\}$ of the elements of $G$, and set $C = \max (\|f\|_\infty, M)$. Given $\epsilon > 0$ find $N$ such that

$$\sup \left\{ \sum_{i=N+1}^\infty |A_i(z)|^2 : z \in \mathfrak{A} \right\} < \epsilon/4C\delta\|F\|_\infty$$

and find $r_0$ such that for $r_0 < r < 1$

$$\sup \left\{ \left| f(rz) - f(z) \right| : z \in \bigcup_{i=1}^N A_i(\mathfrak{A}) \right\} < \epsilon/2C\delta\|F\|_\infty.$$

Then for $z \in \mathfrak{A}$ and $r_0 < r < 1$,

$$|P(f - f_r)(z)| \leq \|F\|_\infty \delta \sum_{i=1}^N \left| f(A_i z) - f_r(A_i z) \right| \cdot |A_i(z)|^2$$

$$+ 2C\|F\|_\infty \delta \sum_{i=N+1}^\infty \left| A_i(z) \right|^2 < \epsilon.$$

For another proof of Corollary 3, see [2, p. 291].
5. Remarks. A closer investigation of Forelli's projection of $H^\omega(\Delta)$ onto $H^\omega(G)$ shows that it too has property (6) but does not extend to $A(\Delta)$. Our projection $P$, like Forelli's projection, has the property that if $f \in L^p$ on $\{|z| = 1\}$, $p \geq 1$ then so is $Pf$. In the present case this fact is a simple consequence of the Hölder inequality and the convergence of $\Sigma |A'(z)|$ for $z \in \bar{G}$. In addition, $P$ obviously has the following property. If $A_p(\Delta)$, $p \geq 1$, is the Banach space of analytic functions in $\Delta$ with norm $\int f^p dx dy < \infty$ and $A_p(G)$ is the subspace of functions satisfying (3), then $P$ is a bounded projection of each $A_p(\Delta)$ onto $A_p(G)$.

References


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