PROJECTIONS TO AUTOMORPHIC FUNCTIONS

C. J. EARLE AND A. MARDEN

1. The main results. Let \( \overline{R} \) be a compact bordered Riemann surface with interior \( R \). We represent \( R \) as the orbit space \( \Delta/G \) where \( G \) is a finitely generated Fuchsian group of the second kind acting on the unit disk \( \Delta \). Choose a fundamental polygon \( \mathcal{F} \) for \( G \) in \( \Delta \) whose closure \( \operatorname{cl}(\mathcal{F}) \) in the plane meets the boundary of \( \Delta \) in a finite number of arcs, each of which corresponds to a boundary contour of \( R \).

On the set of analytic functions in \( \Delta \) we will consider the norms

\[
\|f\|_\infty = \sup\{ |f(z)| : z \in \Delta \},
\]

\[
\|f\| = \iint_{\Delta} |f(z)| \, dx \, dy,
\]

and the corresponding Banach spaces

\[ H^\infty(\Delta) = \{ f : \|f\|_\infty < \infty \} \quad \text{and} \quad A(\Delta) = \{ f : \|f\| < \infty \}. \]

We shall also consider the subspaces \( H^\infty(G) \subset H^\infty(\Delta) \) and \( A(G) \subset A(\Delta) \) of functions which satisfy

\[
f(Az) = f(z) \quad \text{for all } A \in G \quad \text{and} \quad z \in \Delta.
\]

If \( f \) satisfies (3), then

\[
\iint_{\Delta} |f(z)| \, dx \, dy = \iint_{\Delta} |f(z)| \left( \sum_{A \in G} |A'(z)|^2 \right) dx \, dy
\]

so that \( A(G) \) consists of those analytic functions which satisfy (3) and are summable over \( \mathcal{F} \) with respect to the measure

\[
dm(z) = \sum_{A \in G} |A'(z)|^2 dx \, dy.
\]

Thus \( H^\infty(G) \) corresponds to the space of bounded analytic functions on \( R \), and \( A(G) \) to the space of analytic functions on \( R \) which are summable with respect to \( dm \).

Theorem. There is a projection \( P \), bounded in the respective norms, which sends \( A(\Delta) \) (resp. \( H^\infty(\Delta) \)) onto \( A(G) \) (resp. \( H^\infty(G) \)) and has the following property: if \( f(\xi) = f(A\xi) \) for some \( \xi \in \Delta \) and all \( A \in G \), \( f \in A(\Delta) \), then for any \( g \in A(\Delta) \)

Received by the editors December 15, 1966.

1 This research was supported by the National Science Foundation under grants GP 6145 and GP 3904.
In particular, if $f \in A(G)$,

$$P(fg) = fPg.$$  

This Theorem is a direct consequence of the following.

**Lemma.** There is a polynomial $F(z)$ such that the Poincaré series

$$\Theta F(z) = \sum_{A \in G} F(Az)A'(z)^2$$

is bounded away from zero in the fundamental polygon $R$.

A somewhat less general form of the Theorem is due to Forelli [6] who obtained a bounded projection $P$ of $H^\infty(\Delta)$ onto $H^\infty(G)$ having property (7).

2. **Proof of the Lemma.** The set of limit points of $G$ is a closed subset of the unit circle of linear measure zero. If $\Omega$ is the complement of the set of limit points in the extended plane, then $\Omega/G$ is a compact Riemann surface, the double of $R$. Let $R^* \supset \overline{R}$ be a subsurface of $\Omega/G$ such that (i) $R^*$ is bounded by analytic curves, (ii) each component of $R^* - \overline{R}$ is a topological annulus, and (iii) $\pi(\infty)$ is in the exterior of $R^*$ where $\pi: \Omega \to \Omega/G$ is the natural map. Then $\pi^{-1}(R^*) = D^*$ contains $\Delta$, is invariant under $G$, and is bounded by a Jordan curve which is the union of $\pi^{-1}(\partial R^*)$ and the set of limit points of $G$. Moreover, $R^*$ is a compact subset of $D^*$. Let $\partial^*$ be a fundamental "polygon" for $G$ in $D^*$ ($\partial^*$ can be obtained, for example, by mapping $D^*$ onto $\Delta$).

By Abel's theorem there exists a meromorphic differential $\omega$ on the compact surface $\Omega/G$ which is analytic and nonzero on the closure of $R^*$. The quadratic differential $\omega^2$ can be lifted to $D^*$ to determine an analytic $\phi(z)$ which is nonzero in $D^*$ and satisfies

$$\phi(Az)A'(z)^2 = \phi(z) \quad \text{for all } z \in D^*, \ A \in G.$$  

Furthermore, since $\omega^2$ is analytic in the closure of $R^*$,

$$\int \int_{\partial^*} |\phi(z)| \, dx \, dy < \infty.$$  

We now appeal to a recent theorem of Bers [3] concerning Poincaré series in $D^*$. Let $Q(G)$ denote the Banach space of all functions $\phi(z)$ analytic in $D^*$ which satisfy (9) and (10), the norm being given by (10). Bers has proved [3, Theorem 2] that the Poincaré series (8) defines a continuous map of $A(D^*)$ onto $Q(G)$. (A short proof of this
theorem can be found in [4].) Furthermore, since $D^*$ is a Jordan region, a theorem of O. J. Farrell [5] implies that the polynomials are dense in $A(D^*)$.

Applying these results to our nonzero function $\phi(z)$ in $Q(G)$, we obtain a sequence $\{F_n\}$ of polynomials such that $\Theta F_n \rightarrow \phi$ in $Q(G)$. But convergence in $Q(G)$ implies uniform convergence on compact sets in $D^*$. In particular, $\Theta F_n \rightarrow \phi$ uniformly in $\mathcal{R}$, and for sufficiently large $n$, $\Theta F_n$ is bounded away from zero in $\mathcal{R}$. q.e.d.

3. Proof of the Theorem. Choose a polynomial $F(z)$ in accordance with the Lemma. For $f \in A(\Delta)$ define

$$Pf(z) = (\Theta Ff)(z)/\Theta F(z).$$

Set

$$\delta = \inf \{ |\Theta F(z)| : z \in \mathcal{R} \} > 0,$$

$$M = \sup \left\{ \sum_{A \in G} |A'(z)|^2 : z \in \mathcal{R} \right\} < \infty.$$

Then

$$\int \int_{\mathcal{R}} |\Theta Ff| \, dx \, dy \leq \|F\|_\infty \int \int_{\mathcal{R}} \sum_{A \in G} |f(Az)| |A'(z)|^2 \, dx \, dy$$

$$= \|F\|_\infty \sum_{A \in G} \int \int_{A(\mathcal{R})} |f(z)| \, dx \, dy = \|F\|_\infty \|f\|.$$  \hfill (12)

Hence the series $\Theta Ff$ converges absolutely and uniformly on compact subsets of $\mathcal{R}$ and therefore on compact subsets of $\Delta$. Furthermore by the Lemma, $\Theta F$ is nonzero on $\Delta$. Consequently, $Pf$ is analytic in $\Delta$; $Pf$ obviously satisfies (3) and (6). We also have from (4) and (12) that

$$\|Pf\| = \int \int_{\Delta} |Pf(z)| \, dx \, dy = \int \int_{\mathcal{R}} |Pf(z)| \, dm(z)$$

$$\leq M \int \int_{\mathcal{R}} |Pf(z)| \, dx \, dy \leq M\delta \int \int_{\mathcal{R}} |\Theta Ff| \, dx \, dy$$

$$\leq M\delta \|F\|_\infty \|f\|,$$

and, if $f$ is in $H^\infty(\Delta)$ as well,

$$\|Pf\|_\infty = \sup \{ |Pf(z)| : z \in \Delta \}$$

$$= \sup \{ |Pf(z)| : z \in \mathcal{R} \} \leq M\delta \|F\|_\infty \|f\|_\infty.$$  \hfill (14)

The remainder of the Theorem follows at once. q.e.d.
4. Applications. As Forelli has pointed out [6, Corollary 2], Carleson’s solution of the corona problem for $H^\infty(\Delta)$ and the existence of a projection $P: H^\infty(\Delta) \to H^\infty(G)$ with property (7) yield a solution of the corona problem for the compact bordered surface $R$. We state this consequence of Theorem 1 as

**Corollary 1.** Let $\mathfrak{M}(R)$ be the maximal ideal space of the algebra $H^\infty(G)$. Then $R$ is dense in $\mathfrak{M}(R)$.

Other proofs of Corollary 1 have been given in [1], [6], [7].

**Corollary 2.** Let $S$ be a set of points in $\Delta$ which is invariant under $G$ and let $\xi(z)$ be a complex valued function on $S$ such that $\xi(Az) = \xi(z)$ for all $A \in G$ and $z \in S$. There exists $f \in A(G)$ (resp. $H^\infty(G)$) with $f(z) = \xi(z)$ for all $z \in S$ if and only if there exists $g \in A(\Delta)$ (resp. $H^\infty(\Delta)$) with $g(z) = \xi(z)$, all $z \in S$.

Indeed, if $g(z)$ is given, then $f(z) = Pg(z) = \xi(z)$ for all $z \in S$.

Corollary 2 strengthens Stout’s theorem [7] that if $S$ is a $G$-invariant interpolating set for $H^\infty(\Delta)$ it is also an interpolating set for $H^\infty(G)$.

Let $\alpha(G) \subset H^\infty(G)$ be the subspace of functions continuous in $\bar{\Delta}$, and $\alpha_0(G) \subset \alpha(G)$ the subset of functions analytic in $\bar{\Delta}$.

**Corollary 3.** $\alpha_0(G)$ is dense in $\alpha(G)$.

**Proof.** We use the notations of §3. If $f \in \alpha(G)$, set $f_r(z) = f(rz)$ for $r < 1$. Then $P(f_r) \in \alpha_0(G)$. We claim that $P(f_r) \to P(f) = f$ as $r \to 1$. For the proof, choose an enumeration $\{A_i\}$ of the elements of $G$, and set $C = \max (||f||_{\infty}, M)$. Given $\epsilon > 0$ find $N$ such that

$$\sup \left\{ \sum_{i=N+1}^{\infty} |A_i(z)|^2 : z \in \mathfrak{A} \right\} < \epsilon/4C\delta||F||_{\infty}$$

and find $r_0$ such that for $r_0 < r < 1$

$$\sup \left\{ |f(rz) - f(z)| : z \in \bigcup_{i=1}^{N} A_i(\mathfrak{A}) \right\} < \epsilon/2C\delta||F||_{\infty}.$$ 

Then for $z \in \mathfrak{A}$ and $r_0 < r < 1$,

$$|P(f - f_r)(z)| \leq ||F||_{\infty}\delta \sum_{i=1}^{N} |f(A_i z) - f_r(A_i z)| |A_i'(z)|^2$$

$$+ 2C||F||_{\infty}\delta \sum_{i=N+1}^{\infty} |A_i'(z)|^2 < \epsilon.$$

For another proof of Corollary 3, see [2, p. 291].
5. **Remarks.** A closer investigation of Forelli's projection of $H^\infty(\Delta)$ onto $H^\infty(G)$ shows that it too has property (6) but does not extend to $A(\Delta)$. Our projection $P$, like Forelli's projection, has the property that if $f \in L^p$ on $\{|z| = 1\}$, $p \geq 1$ then so is $Pf$. In the present case this fact is a simple consequence of the Hölder inequality and the convergence of $\sum |A'(z)|$ for $z \in \mathbb{R}$. In addition, $P$ obviously has the following property. If $A^p(\Delta)$, $p \geq 1$, is the Banach space of analytic functions in $\Delta$ with norm $\int_\Delta |f(z)|^p \, dx \, dy < \infty$ and $A^p(G)$ is the subspace of functions satisfying (3), then $P$ is a bounded projection of each $A^p(\Delta)$ onto $A^p(G)$.

**References**


**Cornell University and University of Minnesota**