Let $G$ be a compact connected Lie group and $U \subseteq G$ a closed connected subgroup. Our objective is to determine the cohomology of the space of loops on the homogenous space $G/U$. Our main tool is a spectral sequence of Eilenberg and Moore ([3], [4], [7], [8]) from which we obtain the following:

**Theorem P1.** Let $p$ be a prime and $T \subseteq G$ a maximal torus of $G$. Assume that $H^*(G, Z_p)$ is an exterior algebra on odd dimensional generators. Then there exists a filtration $\{ F^{-n}H^*(\Omega(G/T), Z_p); n \geq 0 \}$ such that

$$E^0H^*(\Omega(G/T), Z_p) = \text{Tor}_{H^*(G/T, Z_p)}(Z_p, Z_p)$$

as Hopf algebras.

**Theorem Q1.** Let $Q$ be the rational numbers and let $G, T$ be as in Theorem P1, then there is an isomorphism of algebras

$$H^*(\Omega(G/T), Q) = \text{Tor}_{H^*(G/T, Q)}(Q, Q).$$

**Theorem P2.** Let $G$ be simply connected and $U$ a closed connected subgroup and $i: U \subseteq G$ the inclusion map. Consider $H^*(U, Z_p)$ as an $H^*(G, Z_p)$ module via the map $i^*: H^*(G, Z_p) \rightarrow H^*(U, Z_p)$. Then if $H^*(G, Z_p)$ is an exterior algebra on odd dimensional generators there is a filtration $\{ F^{-n}H^*(\Omega(G/U), Z_p); n \geq 0 \}$ such that

$$E^0H^*(\Omega(G/U), Z_p) = \text{Tor}_{H^*(G, Z_p)}(Z_p, H^*(U, Z_p))$$

as Hopf algebras.

**Theorem Q2.** Let $Q$ be the rational numbers, $G, U$ as in Theorem P2. Then there is an isomorphism of algebras

$$H^*(\Omega(G/U), Q) = \text{Tor}_{H^*(G, Q)}(Q, H^*(U, Q)).$$

1. **Algebra.** Throughout this section $k$ will denote a fixed field. All modules will be graded $k$-modules of finite type, all algebras will be connected associative and commutative; $\otimes$ means $\otimes_k$. Our notation will be that of [5].

**Definition.** If $A$ is an algebra, $a_1, \cdots, a_n \in A$, then we say that $a_1, \cdots, a_n$ is an ESP-sequence if $a_i$ is not a zero divisor in the algebra.
\( A/(a_1, \ldots, a_{i-1}) \) for \( i = 1, \ldots, n-1 \). (Here \( (a_1, \ldots, a_{i-1}) \) denotes the ideal generated by \( a_1, \ldots, a_{n-1} \).)

**Definition.** An algebra \( \Lambda \) is called a saturated algebra if it is the quotient of a polynomial algebra \( P[x_1, \ldots, x_n] \) by an ESP-sequence \( r_1, \ldots, r_n \).

It is easy to see that if \( \Lambda \) is a saturated algebra then it can be presented as a quotient of a polynomial algebra \( P[y_1, \ldots, y_m] \) by an ESP-sequence \( s_1, \ldots, s_m \) where no \( s_i \) is indecomposable. We will always assume that our presentations are minimal in this sense.

Our objective in this section is to compute \( \text{Tor}_\Lambda(k, k) \) when \( \Lambda \) is a saturated algebra. To this end let

\[
\Lambda = P[x_1, \ldots, x_n]/(r_1, \ldots, r_n)
\]

be a minimal presentation of \( \Lambda \). Let

\[
\mathcal{E} = \Lambda \otimes E[u_1, \ldots, u_n]
\]

where \( \deg u_i = (-1, \deg x_i) \). We give \( \mathcal{E} \) the structure of a complex by defining a differential \( \partial \) by the formulas

\[
\partial(\lambda \otimes 1) = 0, \quad \partial(\lambda \otimes u_i) = \lambda x_i \otimes 1, \quad i = 1, \ldots, n, \quad \lambda \in \Lambda,
\]

and requiring \( \partial \) to be a derivation of algebras. An easy calculation [10] shows

\[
H^*(\mathcal{E}) = E[v_1, \ldots, v_n], \quad \deg v_i = (-1, \deg r_i).
\]

Now form the algebra

\[
\mathcal{R} = \mathcal{E} \otimes \Gamma[z_1, \ldots, z_n], \quad \deg z_i = (-2, \deg r_i)
\]

and define a derivation \( d \) on \( \mathcal{R} \) as follows: for each \( i \) choose a representative cocycle \( \tilde{v}_i \in \mathcal{E} \) representing \( v_i \in H(\mathcal{E}) \). Now define \( d \) by

\[
d| \mathcal{E} = \partial, \quad d(z_i) = \tilde{v}_i, \quad i = 1, \ldots, n,
\]

and requiring that \( d \) be a derivation of algebras with divided powers. (Note that if the characteristic of \( k \) is zero then \( \Gamma[z_1, \ldots, z_n] = P[z_1, \ldots, z_n] \).) As in [10] we can now show that \( \mathcal{R} \to k \) is a \( \Lambda \)-free resolution of \( k \) and hence

**Proposition 1.1.** \( \text{Tor}_\Lambda(k, k) = E[u_1, \ldots, u_n] \otimes \Gamma[z_1, \ldots, z_n] \) as a Hopf algebra over \( k \). \( \Box \)

2. **Torus subgroups.** Let \( G \) be a compact connected Lie group and \( T \) a maximal torus of \( G \). The inclusion \( T \subset G \) induces a fibre bundle

\[
\rho: G/T \to B_T \to B_G
\]
where $B_T$ and $B_G$ are classifying spaces for $T$ and $G$ respectively. Let $k = \mathbb{Z}_p$ or $\mathbb{Q}$ and assume that $H^*(G, k)$ is an exterior algebra on odd dimensional generators (this is no restriction if $k = \mathbb{Q}$). We then have the fundamental result of Borel [2]:

**Theorem ([2], [1]).** $H^*(B_G, k) = P[y_1, \cdots, y_n] n = \text{rank } G, p^*(y_1), \cdots, p^*(y_n)$ is an ESP-sequence in $H^*(B_T, k)$ and $H^*(G/T, k) = H^*(B_T, k) \otimes p^*$. Since $H^*(B_T, k) = P[x_1, \cdots, x_n] \deg x_i = 2$ it follows that $H^*(G/T, k)$ is a saturated algebra.

**Proof of Theorem P1.** Let $\{E_r, d_r\}$ denote the Eilenberg-Moore spectral sequence [3], [4], [8] of the pathspace fibration

$$niG/T \to PiG/T \to G/T$$

we then have

$$E_r \Rightarrow H^*(\Omega(G/T), Z_p), \quad E_2 = \text{Tor}_{H^*(G/T, Z_p)}(Z_p, Z_p).$$

We will show that $E_2 = E_\infty$ which will complete the proof. From §1 we see that

$$E_2 = E[u_1, \cdots, u_m] \otimes \Gamma[z_1, \cdots, z_m]$$

where $m \leq n$, $\deg u_i = (-1, 2)$, $\deg z_i = (-2, 2t_i), t_i \geq 2$. Now $\{E_r, d_r\}$ is a spectral sequence of Hopf algebras. Let $s$ be the smallest integer $\geq 2$ such that $d_s \neq 0$. Then $E_2 = E_s$. Let $a \in E_2$ of minimal degree such that $d_s(a) \neq 0$. Then we may assume that $a$ is indecomposable, and therefore $\deg a = (-2p^s, 2tp^s), t > 1$. A simple computation shows that $d_s(a)$ must be primitive. Since the total degree of $a$ is even it follows that $d_s(a)$ is a primitive of odd total degree and hence $\deg d_s(a) = (-1, 2)$. Since $\deg d_s = (s, 1-s)$ we see that $s = 2p^s - 1$ and hence we have $\deg d_s(a) = (-1, 2+2(t-1)p^s)$. But $t > 1$ and hence $2+2(t-1)p^s > 2$ which contradicts our previous calculation of $\deg d_s(a)$. Therefore our assumption that $d_s(a) \neq 0$ must be false and hence $E_2 = E_\infty$. □

**Proof of Theorem Q1.** We recall [2] that $H^*(G, Q)$ is an exterior algebra on odd dimensional generators. By the argument above we see the Eilenberg-Moore spectral sequence of the fibration

$$\Omega G/T \to P(G/T) \to G/T$$

with $Q$ coefficients has $E_2 = E_\infty$.

But $E_2 = E[u_1, \cdots, u_m] \otimes P[z_1, \cdots, z_m]$ is a free commutative algebra and hence the extension as algebras is trivial. □
3. A fibration. Let $U$ be a closed connected subgroup of $G$ and $i: U \subseteq G$ the inclusion map. By a standard construction [9, p. 99] we can replace $i$ by an equivalent fibre mapping whose fibre is $\Omega(G/U)$. This gives a multiplicative fibre map

$$\Omega(G/U) \rightarrow U \rightarrow G.$$ 

If $G$ is simply connected then this fibering has for each field $k$ an Eilenberg-Moore spectral sequence $\{ E_r, d_r \}$ with

$$E_r \Rightarrow H^*(\Omega(G/U), k), \quad E_2 = \text{Tor}_{H^*(G, k)}(k, H^*(U, k)).$$

Henceforth we shall assume that $k = \mathbb{Z}_p$ or $\mathbb{Q}$ and that $H^*(G, k)$ is an exterior algebra on odd dimensional generators. From [2] it follows that $H^*(G, k)$ is primitively generated and hence cocommutative. Let $H^*(G, k) \backslash i^*$ denote the kernel in the category of bicommutative Hopf algebras of the map $i^*: H^*(G, k) \rightarrow \text{im } i^*$ (see [6], [7], [8]). Then as in [7], [8] we can show

**Proposition.**

$$\text{Tor}_{H^*(G, k)}(k, H^*(U, k)) = H^*(U, k) / i^* \otimes \text{Tor}_{H^*(G, k)} \backslash i^*(k, k)$$

as Hopf algebras. □

4. The simply connected case. Before proving Theorem P2 we state a preliminary technical result whose proof we postpone until later.

**Proposition.**

$$\ker\{ j^*: H^*(U, \mathbb{Z}_p) \rightarrow H^*(\Omega(G/U), \mathbb{Z}_p) \} = \text{im } i^* \cdot H^*(U, \mathbb{Z}_p).$$

**Proof of Theorem P2.** By the Proposition of §3 we have

$$E_2 = H^*(U, \mathbb{Z}_p) / i^* \otimes \text{Tor}_{H^*(G, \mathbb{Z}_p)} \backslash i^*(Z_p, Z_p).$$

From Borel's structure theorem for Hopf algebras over $\mathbb{Z}_p$ [6, 7.11] we deduce that $H^*(G, \mathbb{Z}_p) \backslash i^* = E[u_1, \ldots, u_m]$, degree $u_i$ odd, and hence we deduce that as Hopf algebras

$$E_2 = H^*(U, \mathbb{Z}_p) / i^* \otimes \Gamma[y_1, \ldots, y_m]$$

where $\deg y_i = (-1, \deg u_i)$. Suppose that $E_2 \neq E_\infty$. A repetition of the argument in Theorem P1 shows that there is an $x \in E_2$ such that $d_r(x) \neq 0 \in E_2^0$. The map $j^*$ is given by the composition

$$H^*(U, \mathbb{Z}_p) \rightarrow H^*(U, \mathbb{Z}_p) / i^* = E_2^0 \rightarrow E_\infty^0 \subset H^*(\Omega(G/U), \mathbb{Z}_p)$$
and hence from the Proposition above it follows that the edge map $\epsilon$ is a monomorphism. But this contradicts the fact that $d_r(x) \neq 0$. Therefore it follows that $E_2 = E_\infty$. □

**Proof of Proposition.** Consider the fibration

$$\Omega(G) \to \Omega(G/U) \to U$$

obtained by replacing $j$ by a fibre map. Let $\{ \hat{E}^r, d^r \}$ denote the homology Serre spectral sequence of this fibration. Consider the diagram

$$\begin{array}{ccc}
\Omega(G) & \to & \Omega(G) \\
\downarrow & & \downarrow \\
\Omega(G/U) & \to & P(G) \\
\downarrow & & \downarrow \\
U & \to & G
\end{array}$$

and recall that we have by [6, 4.4]

1. $H_\ast(G, \mathbb{Z}_p) = E[x_1, \ldots, x_n]$
2. $H_\ast(U, \mathbb{Z}_p) = E[y_1, \ldots, y_t] \otimes M$ where $i_\ast(y_i) = x_i, \quad 1 \leq i \leq t,$
   $i_\ast/M = 0,$
3. $H_\ast(\Omega(G), \mathbb{Z}_p) = P[z_1, \ldots, z_n]$ with $\tau x_i = z_i$ where $\tau$ is the transgression in the path space fibering over $G$. A standard spectral sequence argument now shows that $\hat{E}^\infty = M \otimes P[z_{t+1}, \ldots, z_n]$ and since the map $j_\ast$ is the composition $H_\ast(U, \mathbb{Z}_p) \to \hat{E}^2_{0,*} \to \hat{E}^\infty_{0,*}$ $\subset H_\ast(\Omega(G/U), \mathbb{Z}_p)$ the result follows by duality. □

The proof of Theorem Q2 is an easy exercise and is left to the reader.

In closing we note that if $G$ is simply connected and $T$ is any torus in $G$ then the inclusion map $i: T \subset G$ is null homotopic and hence standard homotopy theory shows that $\Omega(G/T)$ has the homotopy type of $T \times \Omega(G)$, and hence the study of torus subgroups in the simply connected case is trivial.

**References**


Yale University and
Princeton University