STRETCHING PHENOMENA IN MAPPINGS OF SPHERES

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This paper presents an extension of the recent work of R. Olivier [3] on dilatation phenomena in differentiable mappings of spheres.

Let \( S^k = \{(x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1}: \sum_{j=1}^{k+1} x_j^2 = 1\} \) be provided with the usual unit sphere metric \( d \). Let \( f: S^m \to S^n \) be a differentiable mapping and define a dilatation constant \( \delta_f = \max \|f_*(X)\|/\|X\| \), where \( X \) runs over the nonzero tangent vectors of \( S^m \) and where \( f_* \) is the induced mapping on the tangent vectors. Olivier showed that if \( m = n \) and \( f \) has even nonzero degree, or if \( m > n = 2 \) and \( f \) is not homotopic to zero (designated \( f \not\equiv 0 \)), then \( \delta_f \geq 2 \).

We prove a generalization conjectured by Olivier.

**Theorem 1.** If \( m > n \) for any \( n > 0 \) and if \( f \not\equiv 0 \), then \( \delta_f \geq 2 \).

**Proof.** The Borsuk-Ulam theorem [3, p. 266] guarantees the existence of a point \( x \in S^m \) such that \( f(x) = f(-x) \). Assume \( \delta_f < 2 \). Then each meridian in \( S^m \) from \(-x\) to \( x\) is mapped into a loop at \( a = f(x) \) of length less than \( 2\pi \). Hence \(-a\) does not belong to the image \( f(S^m) \) and therefore \( f \not\equiv 0 \).

Let \( \{f\} \) denote the homotopy class of \( f \) in \( \pi_m(S^n) \) and let \( \Sigma: \pi_{m-1}(S^{n-1}) \to \pi_m(S^n) \) be the suspension homomorphism.

**Theorem 2.** Let \( m = 2k > 0 \) and assume that \( 2\{f\} \not\equiv 0 \). Then if \( \{f\} \in \Sigma(\pi_{2k-1}(S^{n-1})) \), \( \delta_f \geq 3 \).

This statement is of interest only for \( k \geq n \) but has the following interesting corollary.

**Corollary 1.** If \( f: S^{2k} \to S^2 \) is a differentiable mapping with \( k > 2 \) and \( 2\{f\} \not\equiv 0 \), then \( \delta_f \geq 3 \).

We first prove a

**Lemma.** Let \( f: S^{2k} \to S^n \) with \( k > 0 \) be any mapping such that \( 2\{f\} \not\equiv 0 \). Then there exists a point \( x \in S^{2k} \) such that \( f(x) = -f(-x) \).

**Proof.** Suppose that no such point exists. Then the mapping \( \phi: S^{2k} \to S^n \), where \( \phi(x) = (f(x) - f(-x)) / \|f(x) - f(-x)\| \) is well-defined. (The operations are done in \( \mathbb{R}^{n+1} \).) Clearly, \( \phi \cong f \) and \( \phi \cdot A = \phi \).

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where $A : S^{2k} \to S^{2k}$ is the antipodal map. Hence the following triangle commutes.

$$
\begin{array}{ccc}
\pi_{2k}(S^{2k}) & \xrightarrow{A_*} & \pi_{2k}(S^{2k}) \\
\phi_* & \downarrow & \phi_* \\
\pi_{2k}(S^n) & \xrightarrow{\phi_*} & \pi_{2k}(S^n)
\end{array}
$$

Setting $1 = \{\text{identity}\} \subseteq \pi_{2k}(S^{2k})$, we obtain $\{f\} = \phi_*(1) = \phi_*(A_*(1)) = \phi_*(-1) = -\{f\}$ and $2\{f\} = 0$ contrary to assumption.

**Proof of Theorem 2.** Suppose that $2\{f\} \neq 0$ but that $\delta_f < 3$. By the Freudenthal theorems and Theorem 1, it suffices to consider $k \geq n$ and $\delta_f \geq 2$. Let $\lambda$ be a number such that $0 < \lambda < \pi$ and $\delta_f < \frac{2 + \lambda}{\pi}$. By the lemma there is a point $x$ in $S^n$ such that $d(f(x), f(-x)) = \pi$; and by Borsuk-Ulam [3, p. 266], there is a point $x'$ such that $d(f(x'), f(-x')) = 0$. Thus by the continuity of $d$ there is a point $y \in S^m$ such that $d(f(y), f(-y)) = \lambda$.

Denote by $\Omega \cdot S^i(a, b)$ the set of all piecewise smooth paths $\gamma : [0, 1] \to S^i$ such that $\gamma(0) = a$, $\gamma(1) = b$ and $\int_0^1 \|d\gamma/dt\|^2 dt \leq c^2$, with the compact-open topology. Observe that if $\gamma \in \Omega \cdot S^i(a, b)$, then the length of $\gamma$ is less than or equal to $c$ due to the Schwarz inequality.

By definition of $\lambda$, $\delta_f \pi < 2\pi + \lambda$. Therefore, since $f(y)$ and $f(-y)$ are not conjugate along any geodesic in $S^n$, a standard application of Morse Theory [2, p. 96] shows that

$$
\Omega^y \overset{\text{def}}{=} \Omega^y \cdot S^n(f(y), f(-y))
$$

has the homotopy type of a $C$-$W$ complex with one cell in dimension zero and one cell in dimension $n - 1$, i.e., $\Omega^y$ has the homotopy type of $S^{n-1}$. Observe that in a natural way $f$ induces a mapping $\tilde{f} : \Omega^x \cdot S^{2k}(y, -y) \to \Omega^y$. Clearly, $\Omega^x \cdot S^{2k}(y, -y)$ is homeomorphic to $S^{2k-1}$ and $\tilde{f}$ determines a unique class in $\pi_{2k-1}(S^{n-1})$.

It remains only to show that $\Sigma(\{\tilde{f}\}) = \{f\}$. Let

$$
\Omega S^n = \Omega S^n(f(y), f(-y))
$$

be the full path space of $S^n$ and $i : \Omega^y \subset \Omega S^n$ the inclusion map. Denote by

$$
\theta : \pi_{2k}(S^n) \xrightarrow{\cong} \pi_{2k-1}(\Omega S^n)
$$

the standard adjoint isomorphism. Then the class in $\pi_{n-1}(\Omega S^n)$ deter-
mined by $i$ corresponds to the adjoint of the identity map of $S^n$, and the following diagram commutes.

\[
\begin{array}{c}
\pi_{2k-1}(S^{n-1}) \\
\downarrow i_* \\
\pi_{2k-1}(\Omega S^n)
\end{array} \quad \quad \begin{array}{c}
\theta \\
\Sigma
\end{array} \quad \begin{array}{c}
\pi_{2k}(S^n)
\end{array}
\]

From the construction it is clear that $i_*(\{f\}) = \theta(\{f\})$, and thus $\Sigma(\{f\}) = \{f\}$. This completes the proof.

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**BIBLIOGRAPHY**


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