STRETCHING PHENOMENA IN MAPPINGS OF SPHERES[1]

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This paper presents an extension of the recent work of R. Olivier [3] on dilatation phenomena in differentiable mappings of spheres.

Let $S^k = \{(x_1, \cdots, x_{k+1}) \in R^{k+1}: \sum_{j=1}^{k+1} x_j^2 = 1\}$ be provided with the usual unit sphere metric d. Let $f: S^m \to S^n$ be a differentiable mapping and define a dilatation constant $\delta_f = \max \|f_*(X)\|/\|X\|$, where X runs over the nonzero tangent vectors of S^m and where f_* is the induced mapping on the tangent vectors. Olivier showed that if $m=n$ and f has even nonzero degree, or if $m>n=2$ and f is not homotopic to zero (designated $f \not\simeq 0$), then $\delta_f \geqq 2$.

We prove a generalization conjectured by Olivier.

THEOREM 1. *If $m>n$ for any $n>0$ and if $f \not\simeq 0$, then $\delta_f \geqq 2$.*

PROOF. The Borsuk-Ulam theorem [3, p. 266] guarantees the existence of a point $x \in S^m$ such that $f(x) = f(-x)$. Assume $\delta_f < 2$. Then each meridian in S^m from $-x$ to x is mapped into a loop at $a = f(x)$ of length less than 2π. Hence $-a$ does not belong to the image $f(S^m)$ and therefore $f \simeq 0$.

Let $\{f\}$ denote the homotopy class of f in $\pi_m(S^n)$ and let $\Sigma: \pi_{m-1}(S^{n-1}) \to \pi_m(S^n)$ be the suspension homomorphism.

THEOREM 2. *Let $m=2k>0$ and assume that $2\{f\} \neq 0$. Then if $\{f\} \notin \Sigma(\pi_{2k-1}(S^{n-1}))$, $\delta_f \geqq 3$.*

This statement is of interest only for $k \geqq n$ but has the following interesting corollary.

COROLLARY 1. *If $f: S^{2k} \to S^2$ is a differentiable mapping with $k>2$ and $2\{f\} \neq 0$, then $\delta_f \geqq 3$.*

We first prove a

LEMMA. *Let $f: S^{2k} \to S^n$ with $k>0$ b e any mapping such that $2\{f\} \neq 0$ Then there exists a point $x \in S^{2k}$ such that $f(x) = -f(-x)$.*

PROOF. Suppose that no such point exists. Then the mapping $\phi: S^{2k} \to S^n$, where $\phi(x) = (f(x) - f(-x))/|f(x) - f(-x)|$ is well defined. (The operations are done in R^{n+1}.) Clearly, $\phi \simeq f$ and $\phi \cdot A = \phi$,

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where $A: S^{2k} \to S^{2k}$ is the antipodal map. Hence the following triangle commutes.

$$\begin{tikzcd} \pi_{2k}(S^{2k}) \arrow[rr, "A_*"] \arrow[dr, "\phi_*"'] & & \pi_{2k}(S^{2k}) \arrow[dl, "\phi_*"] \\ & \pi_{2k}(S^n) & \end{tikzcd}$$

Setting $1 = \{\text{identity}\} \in \pi_{2k}(S^{2k})$, we obtain $\{f\} = \phi_*(1) = \phi_*(A_*(1)) = \phi_*(-1) = -\{f\}$ and $2\{f\} = 0$ contrary to assumption.

Proof of Theorem 2. Suppose that $2\{f\} \neq 0$ but that $\delta_f < 3$. By the Freudenthal theorems and Theorem 1, it suffices to consider $k \geqq n$ and $\delta_f \geqq 2$. Let λ be a number such that $0 < \lambda < \pi$ and $\delta_f < 2 + \lambda/\pi$. By the lemma there is a point x in S^m such that $d(f(x), f(-x)) = \pi$; and by Borsuk-Ulam [**3**, p. 266], there is a point x' such that $d(f(x'), f(-x')) = 0$. Thus by the continuity of d there is a point $y \in S^m$ such that $d(f(y), f(-y)) = \lambda$.

Denote by $\Omega^c S^l(a, b)$ the set of all piecewise smooth paths $\gamma: [0, 1] \to S^l$ such that $\gamma(0) = a$, $\gamma(1) = b$ and $\int_0^1 \|d\gamma/dt\|^2 dt \leqq c^2$, with the compact-open topology. Observe that if $\gamma \in \Omega^c S^l(a, b)$, then the length of γ is less than or equal to c due to the Schwarz inequality.

By definition of λ, $\delta_f \pi < 2\pi + \lambda$. Therefore, since $f(y)$ and $f(-y)$ are not conjugate along any geodesic in S^n, a standard application of Morse Theory [**2**, p. 96] shows that

$$\Omega^{\delta_f} \overset{\text{def.}}{=} \Omega^{\delta_f \pi} S^n(f(y), f(-y))$$

has the homotopy type of a C-W complex with one cell in dimension zero and one cell in dimension $n-1$, i.e., Ω^{δ_f} has the homotopy type of S^{n-1}. Observe that in a natural way f induces a mapping $\bar{f}: \Omega^\pi S^{2k}(y, -y) \to \Omega^{\delta_f}$. Clearly, $\Omega^\pi S^{2k}(y, -y)$ is homeomorphic to S^{2k-1} and $\bar{f}$ determines a unique class in $\pi_{2k-1}(S^{n-1})$.

It remains only to show that $\Sigma(\{\bar{f}\}) = \{f\}$. Let

$$\Omega S^n = \Omega S^n(f(y), f(-y))$$

be the full path space of S^n and $i: \Omega^{\delta_f} \subset \Omega S^n$ the inclusion map. Denote by

$$\theta: \quad \pi_{2k}(S^n) \xrightarrow{\approx} \pi_{2k-1}(\Omega S^n)$$

the standard adjoint isomorphism. Then the class in $\pi_{n-1}(\Omega S^n)$ deter-

mined by i corresponds to the adjoint of the identity map of S^n, and the following diagram commutes.

$$\begin{tikzcd}
 & \pi_{2k}(S^n) \arrow[dd, "\theta"] \\
\pi_{2k-1}(S^{n-1}) \arrow[ur, "\Sigma"] \arrow[dr, "i_*"'] & \\
 & \pi_{2k-1}(\Omega S^n)
\end{tikzcd}$$

From the construction it is clear that $i_*(\{\bar{f}\})=\theta(\{f\})$, and thus $\Sigma(\{\bar{f}\})=\{f\}$. This completes the proof.

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