ON AN INEQUALITY CONSIDERED BY ROBERTSON

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Let S denote the usual family of normalized univalent functions $f(z)$ in the unit circle whose power series expansion about the origin is given by

$$f(z) = z + \sum_{n=2}^{\infty} A_n z^n \tag{1}$$

where we write also for convenience $A_1 = 1$. Robertson [2] has recently considered an inequality involving the coefficients of (1) which if valid for the family S would imply the truth of the Bieberbach conjecture. He verified that this inequality

$$\left| m|A_m| - n|A_n| \right| \leqq |m^2 - n^2|, \qquad m, n = 1, 2, \cdots, \tag{2}$$

did hold for several special subclasses of S. He made no conjecture concerning its general validity for S. Indeed, it is not in general valid and in particular fails for $m=3$, $n=2$. In this case the inequality would be $\left|3|A_3| - 2|A_2|\right| \leqq 5$. Our theorem will be stated in the following form.

THEOREM. *For the coefficients of the function* (1) *there hold the inequalities*

$$-2 \leqq 3|A_3| - 2|A_2| \leqq \tfrac{4}{3} - \tfrac{1}{2}\tau_0 + \tfrac{3}{8}\tau_0^2.$$

The lower bound is attained only for the functions $z(1+e^{i\phi}z+e^{2i\phi}z^2)^{-1}$, *$\phi$ real. The upper bound is attained only for the functions* $[f(z^{-1}, \tau_0, \phi)]^{-1}$ *as defined in* [1, *p.* 171] *with ϕ real and τ_0 the larger root of the equation*

$$\tfrac{3}{2}\tau \log(\tau/4) + 1 = 0$$

for $\tau > 0$. *This upper bound is in particular greater than* 5.

The lower bound is elementary. In particular $3|A_3| - 2|A_2| > -2$ if $|A_2| < 1$ while for $1 \leqq |A_2| \leqq 2$ we have from the familiar result $|A_3 - A_2^2| \leqq 1$ the inequality

$$\begin{aligned} 3|A_3| - 2|A_2| &\geqq 3|A_3| - 3|A_2|^2 + 3|A_2|^2 - 2|A_2| \\ &\geqq -3 + \min_{1 \leqq x \leqq 2} (3x^2 - 2x) \geqq -2. \end{aligned}$$

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Equality is possible only if both $|A_2|=1$ and $|A_3-A_2^2|=1$, which requires a function of the form prescribed in our theorem.

For the upper bound we recall the result of Corollary 5 in [1] by which if $|A_2|=0$, $|A_3|\leqq 1$ while if

$$|A_2| \leqq \tfrac{1}{2}\tau(1-\log(\tau/4))$$

with $0<\tau\leqq 4$, we have

$$|A_3| \leqq 1+\tfrac{1}{8}\tau^2-\tfrac{1}{4}\tau^2\log(\tau/4)+\tfrac{1}{4}\tau^2(\log(\tau/4))^2.$$

The first eventuality evidently does not correspond to the maximum and we have $3|A_3|-2|A_2| \leqq \max_{0\leqq\tau\leqq 4}\Psi(\tau)$ where

$$\begin{aligned}\Psi(\tau) &= 3, \qquad \tau=0,\\ &= 3+\tfrac{3}{8}\tau^2-\tfrac{3}{4}\tau^2\log(\tau/4)+\tfrac{3}{4}\tau^2(\log(\tau/4))^2\\ &\quad -\tau(1-\log(\tau/4)), \qquad 0<\tau\leqq 4.\end{aligned}$$

A direct calculation gives for $0<\tau\leqq 4$

$$d\Psi(\tau)/d\tau = \log(\tau/4)((3/2)\tau\log(\tau/4)+1).$$

To find the maximum of $\Psi(\tau)$ we observe the function $F(\tau)=(3/2)\tau\ \log(\tau/4)+1$. Its derivative with respect to τ is $(3/2)(\log(\tau/4)+1)$. Thus, on the interval $[0, 4]$, $F(\tau)$ starts with the value 1 at $\tau=0$ (defined by continuity), decreases to the point $\tau=4e^{-1}$ at which $F(\tau)$ is negative, then increases to the value 1 at $\tau=4$. In particular, $F(\tau)$ has two zeros τ_0', τ_0 on $(0, 4)$, $\tau_0'<\tau_0$. Correspondingly, $\Psi(\tau)$ starts with the value 3 at $\tau=0$, decreases to the point $\tau=\tau_0'$, increases to the point $\tau=\tau_0$, then decreases to the value 5 at $\tau=4$. The maximum of $\Psi(\tau)$ on $[0, 4]$ evidently does not occur for $\tau=0$, thus it does occur for $\tau=\tau_0$, and this maximum exceeds 5. One could readily find an explicit numerical approximation for $\Psi(\tau_0)=8/3-\frac{1}{2}\tau_0+\frac{3}{8}\tau_0^2$. The equality statement of our theorem follows at once from the corresponding statement in [1, Corollary 5].

Since the functions used to provide this counterexample to the validity of Robertson's inequality for the family S are those used to prove $|A_3|\leqq 3$ in [1], this situation seems to shed little light on the validity of the Bieberbach conjecture.

Bibliography

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