A COMPATIBILITY THEOREM FOR TWO POINT BOUNDARY PROBLEMS

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Let P be an n by n matrix whose elements are complex valued functions continuous in (t, λ) for t in the closed interval $[a, b]$ and λ in the closed disk $R=\{\lambda: |\lambda-\lambda_0| \leqq r, r>0\}$ in the complex plane. Consider the n-vector differential equation

$$(1) \qquad dy/dt = Py$$

together with the homogeneous n-vector two point boundary conditions

$$(2) \qquad Uy \equiv Ay(a) + By(b) = 0,$$

where A and B are n by n matrices the elements of which are continuous complex valued functions on R.

The compatibility of (1), (2) at $\lambda \in R$ is defined to be the maximum number of linearly independent solutions of (1), (2) corresponding to λ. Let $y_1, \cdots, y_n$ be a fundamental set of solutions of (1) continuous in (t, λ) for $t \in [a, b]$ and $\lambda \in R$, hence uniformly continuous there. A necessary and sufficient condition for the compatibility of (1), (2) at λ to be k is that the rank of the n by n matrix $V(\lambda)$ with elements $U_i y_j(\lambda)$ be of rank $n-k$. It is known [**1**], [**2**] that if the compatibility of (1), (2) is constant in some neighborhood of λ_0 and $x(t)$ is a solution of (1), (2) for $\lambda=\lambda_0$, then there exists a solution $x(t, \lambda)$ of (1), (2) which is uniformly continuous in (t, λ) for $t \in [a, b]$ and λ in some neighborhood of λ_0 and which is such that $x(t, \lambda_0)=x(t)$ so that $x(t, \lambda) \rightarrow x(t)$ uniformly on $[a, b]$. The question, a natural one, as to what can be said in case the compatibility is not constant in any neighborhood of λ_0 was brought to the attention of the author by W. M. Whyburn [**3**]. An answer to the question is in the following theorem.

THEOREM 1. *If the compatibility of* (1), (2) *at* λ_0 *is* $k_0>0$ *and is not identically zero in any deleted neighborhood of* λ_0, *then there exists an integer* k *with* $0<k\leqq k_0$, *an infinite sequence* $\{\lambda_m\}$ *of distinct values of* λ *converging to* λ_0 *at each* λ_m *of which the compatibility of* (1), (2) *is* k, *and correspondingly* k *linearly independent sequences* $\{x_1(t, \lambda_m)\}$, $\cdots$, $\{x_k(t, \lambda_m)\}$ *of solutions of* (1), (2) *for* $\lambda=\lambda_m$ *which converge uni-*

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formly on $[a, b]$ *to some* k *linearly independent solutions* $x_1(t), \cdots,$ $x_k(t)$ *of* (1), (2) *for* $\lambda = \lambda_0$.

PROOF. $V(\lambda)$ as defined above is continuous on R, implying the compatibility of (1), (2) is less than or equal to k_0 at each λ in some neighborhood of λ_0. There exists an integer k, $0 < k \leqq k_0$, and an infinite sequence S of distinct values of λ converging to λ_0 at each term of which the compatibility of (1), (2) is k. The case $k = n$ is disposed of as being trivial since in this case the rank of $V(\lambda)$ is zero for $\lambda = \lambda_0$ and every $\lambda \in S$, implying $y_1, \cdots, y_n$ are linearly independent solutions of (1), (2) for $\lambda = \lambda_0$ and every $\lambda \in S$. For the case $k < n$ then, since $V(\lambda)$ is n by n with n finite, it is possible to select an infinite subsequence $\{\lambda_m\}$ of S together with two finite subsequences $i_1, \cdots, i_{n-k}$ and $j_1, \cdots, j_{n-k}$ of the finite sequence $1, \cdots, n$ which for every m satisfies (i) the $n-k$ by $n-k$ matrix $\delta_m = [U_{i_\alpha} y_{j_\beta}(\lambda_m)]$ is such that $\det \delta_m \neq 0$ and (ii) if $h_1, \cdots, h_{n-k}$ is any subsequence of $1, \cdots, n$, then the $n-k$ by $n-k$ matrix $\Delta_m = [U_{i_\alpha} y_{h_\beta}(\lambda_m)]$ is such that $|\det \Delta_m| \leqq |\det \delta_m|$ and, moreover, $\lim \det \Delta_m / \det \delta_m$ exists, in fact is bounded by one. As a matter of convenience it is assumed that $i_1 = j_1 = 1, \cdots, i_{n-k} = j_{n-k} = n-k$ so that $\delta_m = [U_i y_j(\lambda_m)]$, $i, j = 1, \cdots, n-k$. Now, with $y_j = (y_{1j}, \cdots, y_{nj})$, consider the k infinite sequences of vectors $\{x_j(t, \lambda_m)\}$, $j = 1, \cdots, k$, where for each j the n components $x_{1j}, \cdots, x_{nj}$ of x_j are determinants of order $n-k+1$ defined by

$$x_{ij}(t, \lambda_m) = \begin{vmatrix} y_{i1}(t, \lambda_m) & \cdots\, y_{in-k}(t, \lambda_m) & c_m y_{in-k+j}(t, \lambda_m) \\ U_1 y_1(\lambda_m) & \cdots\, U_1 y_{n-k}(\lambda_m) & c_m U_1 y_{n-k+j}(\lambda_m) \\ \cdot & \cdots \quad \cdot & \cdot \\ U_{n-k} y_1(\lambda_m) & \cdots\, U_{n-k} y_{n-k}(\lambda_m) & c_m U_{n-k} y_{n-k+j}(\lambda_m) \end{vmatrix}$$

with $c_m = (-1)^{n-k}/\det \delta_m$. Clearly, $x_1(t, \lambda_m), \cdots, x_k(t, \lambda_m)$ are linearly independent solutions of (1), (2) for $\lambda = \lambda_m$ inasmuch as each x_j is a linear combination of $y_1, \cdots, y_{n-k}, y_{n-k+j}$ with the coefficient of y_{n-k+j} equal to one. Moreover, $\lim x_j(t, \lambda_m)$ exists; indeed, because

$$\begin{aligned} x_j(t) &= \lim x_j(t, \lambda_m) \\ &= \gamma_{j1} y_1(t, \lambda_0) + \cdots + \gamma_{jn-k} y_{n-k}(t, \lambda_0) + y_{n-k+j}(t, \lambda_0) \end{aligned}$$

for some $\gamma_{j1}, \cdots, \gamma_{jn-k}$, it follows that $x_1(t), \cdots, x_k(t)$ are linearly independent solutions of (1), (2) for $\lambda = \lambda_0$ and that the convergence is uniform on $[a, b]$, completing the proof.

Let $p_1, \cdots, p_n$ be complex valued functions continuous in (t, λ) for $t \in [a, b]$ and $\lambda \in R$. Theorem 2 below concerns itself with the nth order linear scalar differential equation

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = 0 \tag{3}$$

subject to (2). Its proof reflects a simple application of Theorem 1.

THEOREM 2. *Suppose that* $\int_a^b u(t)\bar{v}(t)dt = 0$ *for every pair of solutions* u, v *of* (3), (2) *corresponding to distinct values of* λ. *Then there exists a deleted neighborhood of* λ_0 *at each* λ *of which the compatibility of* (3), (2) *is zero.*

PROOF. If the theorem is false, Theorem 1 then guarantees the existence of an infinite sequence $\{\lambda_m\}$ of distinct values of λ converging to λ_0 and a sequence $\{x_m\}$ of solutions of (3), (2) for $\lambda = \lambda_m$ converging uniformly on $[a, b]$ to x, a solution of (3), (2) for $\lambda = \lambda_0$ which is not the trivial solution. But $\int_a^b x\bar{x}_m = 0$ for every m together with uniform convergence implies $\int_a^b x\bar{x} = 0$, an impossibility.

COROLLARY. *The eigenvalues of a regular selfadjoint problem on a finite interval are isolated.*

REFERENCES

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