SYSTEMS OF TOEPLITZ OPERATORS ON H^2

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1. **Introduction.** Let L^p $(0<p\leqq\infty)$ be the usual Lebesgue space with respect to normalized Lebesgue measure on the unit circle. The space H^p $(0<p\leqq\infty)$ will consist of analytic functions f on the unit disc such that $\lim_{r\to 1-}\|f(re^{i\theta})\|_p<\infty$. If $f\in H^p$, then the function defined a.e. by $f(e^{i\theta})=\lim_{r\to 1-}f(re^{i\theta})$ is in L^p. Should $p\geqq 1$, such functions in L^p are precisely those which have vanishing negative Fourier coefficients. The subspace of functions $f\in H^p$ such that $f(0)=0$ is denoted by H_0^p.

The space of $n\times m$-dimensional, matrix-valued functions whose terms are in L^p will be denoted by $L^p_{n\times m}$. The spaces $H^p_{n\times m}$ and $H^p_{0;n\times m}$ are defined analogously.

Let $\phi\in L^\infty_{n\times n}$. Define the operator T_ϕ: $H^2_{n\times 1}\to H^2_{n\times 1}$ by setting $T_\phi f=P\phi f$ for all $f\in H^2_{n\times 1}$ where P: $L^2_{n\times m}\to H^2_{n\times m}$ is the projection operator. The operator T_ϕ can be considered as a system of Toeplitz operators on H^2, and T_ϕ will be called the *Toeplitz operator* associated with the matrix-valued function ϕ.

This paper concerns conditions on $\phi\in L^\infty_{n\times n}$ which give an invertible T_ϕ. In the scalar case, this problem has been solved by Hartman and Wintner [**3**] whenever ϕ is real valued. In the complex scalar case, necessary and sufficient conditions were given by both Devinatz [**1**] and Widom [**7**]. By using an approach similar to that of Kreĭn and Gohberg [**2**], the factorization requirements shown by Widom are extended to the matrix case. From this it is shown that the problem can be reduced to a consideration of the case where ϕ is unitary.

2. **Definitions and general results.** Most of the preliminary results will be stated without proof. The proofs are either available in the literature, or require only a simple generalization of proofs in the literature. See, for example, Helson [**4**] and Wiener and Masani [**8**].

Let $d\mu$ denote normalized Lebesgue measure on the unit circle.

DEFINITION 2.1. An analytic scalar function f on the unit disc is *outer* provided

$$f(z)=\alpha\exp\left[\int k(e^{i\theta})\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu\right]$$

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[1] The results of this paper are contained in the author's doctoral dissertation, University of Virginia, 1967.

where k is real-valued summable on the unit circle and α is a complex constant of unit modulus.

An analytic function g is *inner* provided $|g(z)| \leqq 1$ inside the unit disc and $|g(e^{i\theta})| = 1$ a.e.

It is readily shown that $f \in H^p$ is outer provided f has the form $f = \alpha \exp[u + iv]$ where u and v are real, $u + iv$ is analytic in the unit disc, u and $\exp(pu)$ are summable, and α is a constant of unit modulus. Thus f is outer in H^p if and only if f^p is outer in H^1; and f outer, $f^{-1} \in L^p$ implies $f^{-1} \in H^p$. The theorem which follows is proved by Helson [**4**] for $f \in H^2$. The proof of the following remains essentially the same.

THEOREM 2.2. *Let $f \in H^p$, $f \neq 0$ a.e. Then f is outer if and only if $\int \log|f| \, d\mu = \log|f(0)| > -\infty$.*

For F a matrix function, let $F_k = \int F(e^{i\theta}) e^{-ik\theta} d\mu$; $k = 0, \pm 1, \pm 2, \cdots$. Denote the conjugate transpose, transpose, and complex conjugate of F by F^*, F', and $\overline{F}$ respectively. A matrix J will be called a projection matrix provided $J = J^2 = J^*$.

DEFINITION 2.3. A function $F \in H^p_{n \times n}$ is *outer* provided $|\det F_0| > 0$ and $\int \log|\det F| \, d\mu = \log|\det F_0|$. A function $F \in H^\infty_{n \times n}$ is *inner* provided $F = JU$ where J is a constant projection matrix and U is unitary a.e.

Thus $F \in H^p_{n \times n}$ is outer if and only if $\det F \in H^{p/n}$ is outer.

THEOREM 2.4. *If $F \in H^2_{n \times n}$ is outer, and if $F^{-1} \in L^2_{n \times n}$, then $F^{-1} \in H^2_{n \times n}$.*

THEOREM 2.5. *If F, $F^{-1} \in H^2_{n \times n}$, then F is an outer function.*

PROOF. Since F, $F^{-1} \in L^2_{n \times n}$, it follows that $-\infty < \int \log|\det F| \, d\mu < \infty$. Hence $\int \log(\det FF^*) d\mu > -\infty$, and the positive semidefinite hermitian function $FF^* \in L^1_{n \times n}$ has a factorization of the form $FF^* = BB^*$ where $B \in H^2_{n \times n}$ is outer, see [**5**, p. 193]. Since $F^{-1} \in H^1_{n \times n}$, $B^{-1} \in L^2_{n \times n}$, and 2.4 implies $B^{-1} \in H^2_{n \times n}$. Also $B^{-1}F$, $(B^{-1}F)^* \in H^2_{n \times n}$ implies $B^{-1}F$ is a constant matrix and a simple check shows it to be unitary. Since $F = B(B^{-1}F)$, F is an outer function.

Let $\Delta_{n \times n}$ denote the space of analytic trigonometric polynomials. For $F \in H^p_{n \times n}$, $S(F)$ will be the subspace of $H^p_{n \times n}$ spanned by functions of the form FP, P varying throughout $\Delta_{n \times n}$.

For a proof of the following see Masani [**6**, p. 286].

THEOREM 2.6. *Let $K \in H^2_{n \times n}$; $K \neq 0$. Then $K = FG$ where $F \in H^2_{n \times n}$ is outer and $G \in H^\infty_{n \times n}$ is inner. Also $S(G) = S(K)$.*

For all $F\in L^2_{n\times n}$, take $\|F\|=(\int \mathrm{tr}\ FF^*d\mu)^{1/2}$.

Since it can be shown that the Hilbert space adjoint T_ϕ^* of T_ϕ is T_{ϕ^*}, it follows that T_ϕ is invertible if and only if T_{ϕ^*} is invertible.

3. **Factorization.** Suppose $\phi\in L^\infty_{n\times n}$ is such that T_ϕ is invertible. There exists a unique $F\in H^2_{n\times n}$ such that

$$T_\phi F = P\phi F = I. \tag{3.1}$$

Thus $\phi F=G^*$ where $G\in H^2_{n\times n}$ and $G_0=I$.

LEMMA 3.2. *If $f\in H^2_{n\times 1}$ and if $\phi f=\bar{g}$ where $g\in H^2_{n\times 1}$, then $f_0\neq 0$.*

PROOF. If $f_0=0$, then $P\phi(e^{-i\theta}f)=0$ and T_ϕ cannot be invertible.

LEMMA 3.3. *If $F\in H^2_{n\times n}$ is the solution of equation* (3.1), *then* $\det F_0\neq 0$.

PROOF. Suppose $\det F_0=0$. Then there exists a linear combination f of the column vectors of F such that $f_0=0$ and $\phi f=\bar{g}$ where $g\in H^2_{n\times 1}$ which contradicts Lemma 3.2.

THEOREM 3.4. *In order that T_ϕ be invertible, it is necessary and sufficient that*

(i) $\phi=G^*H$ where G, G^{-1}, H, $H^{-1}\in H^2_{n\times n}$; and

(ii) K: $f\to H^{-1}P(G^{-1})^*f$ defines a bounded operator from $L^2_{n\times 1}$ to $H^2_{n\times 1}$.

PROOF. Suppose (i) and (ii) hold. For all $f\in H^\infty_{n\times 1}$,

$$P\phi H^{-1}P(G^{-1})^*f = Pf - PG^*(I-P)(G^{-1})^*f = Pf = f;$$

That is, $T_\phi Kf=f$. It follows that $T_\phi Kf=f$ for all $f\in H^2_{n\times 1}$. Similarly, for all $f\in H^\infty_{n\times 1}$,

$$H^{-1}P(G^{-1})^*P\phi f=H^{-1}P(G^{-1})^*\phi f-H^{-1}P(G^{-1})^*(I-P)\phi f=H^{-1}PHf=f,$$

so $KT_\phi f=f$ for all $f\in H^2_{n\times 1}$. Hence $T_\phi^{-1}=K$.

Conversely, suppose $\phi\in L^\infty_{n\times n}$ is such that T_ϕ is invertible. Necessarily T_{ϕ^*} is also invertible. Now consider the solutions of equation (3.1) for ϕ and ϕ^*; that is, $\phi R=S^*$ and $\phi^*M=N^*$, respectively, where R, S, M, $N\in H^2_{n\times n}$, $S_0=N_0=I$. Since $M^*\phi=N$, we have $M^*\phi R=R^0$ $=M_0^*$. Now Lemma 3.3 implies R_0 is nonsingular so $R^{-1}=R_0^{-1}M^*\phi$ $=R_0^{-1}N\in H^2_{n\times n}$. Similarly, $M^{-1}\in H^2_{n\times n}$. Let $G=M^{-1}$ and $H=R_0R^{-1}$, then $\phi=G^*H$ where G, G^{-1}, H, $H^{-1}\in H^2_{n\times n}$. The application of Theorem 2.5 followed by Theorem 2.6 gives $S(G)=S(H)=H^2_{n\times n}$.

Let $M_0>0$ be such that $\|T_\phi^{-1}f\| \leqq M_0\|f\|$ for all $f\in H^2_{n\times 1}$. For all $f\in L^2_{n\times 1}$,

$$H^{-1}P(G^{-1})^*f = H^{-1}P(G^{-1})^*Pf + H^{-1}P(G^{-1})^*(I-P)f = H^{-1}P(G^{-1})^*Pf.$$

Hence $\|H^{-1}P(G^{-1})^*f\| = \|H^{-1}P(G^{-1})^*Pf\| \leqq M_0\|Pf\| \leqq M\|f\|$ for some $M>0$ and all $f\in L^2_{n\times 1}$.

COROLLARY 3.5. *If T_ϕ is invertible, then $\phi^{-1}\in L^\infty_{n\times n}$.*

PROOF. Let $\phi=G^*H$, see 3.4. For an arbitrary $F\in L^\infty_{n\times n}$,

$$\lim_{r\to\infty}\|e^{-ir\theta}P(G^{-1})^*Fe^{ir\theta} - (G^{-1})^*F\| = 0.$$

Thus there exists a subsequence which converges pointwise a.e. to $(G^{-1})^*F$, therefore

$$\left| [H^{-1}P(G^{-1})^*Fe^{im\theta}]_{i,j}\right| \to \left| [H^{-1}(G^{-1})^*F]_{i,j}\right| \quad \text{a.e.}$$

for $i, j=1, 2, \cdots, n$. Since $\|H^{-1}P(G^{-1})^*Fe^{im\theta}\| \leqq M\|F\|$ $(M>0)$ Fatou's lemma implies $\|H^{-1}(G^{-1})^*F\| \leqq M\|F\|$. It can be deduced directly from the analogous result for the scalar case that $H^{-1}(G^{-1})^* = \phi^{-1}\in L^\infty_{n\times n}$.

COROLLARY 3.6. *Suppose T_ϕ is selfadjoint. In order that T_ϕ be invertible, it is both necessary and sufficient that*

(i) *$\phi=H^*UH$ where H, $H^{-1}\in H^2_{n\times n}$, and U is constant unitary with* det $U=\pm 1$.

(ii) *K: $f\to H^{-1}P(H^{-1})^*f$ defines a bounded operator from $L^2_{n\times 1}$ to $H^2_{n\times 1}$.*

PROOF. Let $R\in H^2_{n\times n}$ be the solution of equation (3.1). Now $\phi=\phi^*$, so $R^*\phi R=R_0$. Since R_0 is hermitian, there exists a unitary matrix V such that $D=V^*R_0V$ is diagonal. Write $D=ED_+=D_+E$ where E is diagonal with diagonal terms ± 1 and D_+ has positive entries. Let $B=VD_+^{1/2}V^*$ and $U=VEV^*$, then $R_0=BUB$ where U is the required constant unitary matrix and the constant matrix B is positive definite. Set $H=BR^{-1}$, and the conclusion follows.

If T_ϕ is invertible and selfadjoint, then 3.6 implies that sgn det ϕ is constant. This combined with the fact that $\phi^{-1}\in L^\infty_{n\times n}$ implies that $0\notin$ [ess inf det ϕ, ess sup det ϕ].

In the case where ϕ is positive definite, 3.6 readily simplifies to the following: T_ϕ is invertible if and only if ϕ has the form $\phi=H^*H$ where H, $H^{-1}\in H^\infty_{n\times n}$.

NOTE 3.7. Suppose $\phi\in L^\infty_{n\times n}$ is positive definite. The following are equivalent.

(i) T_ϕ is invertible;

(ii) ess inf det $\phi>0$;

(iii) $T_{\det \phi}$ is invertible.

The equivalence of (ii) and (iii) follows from a result of Hartman and Wintner [3]; which is, if the scalar function ϕ is real, bounded and measurable, then the spectrum of T_ϕ equals [ess inf ϕ, ess sup ϕ].

LEMMA 3.8. *If $F\in H^\infty_{n\times n}$, then $T_\phi T_F=T_{\phi F}$ and $T_{F^*}T_\phi=T_{F^*\phi}$.*

PROOF. For all $f\in H^2_{n\times n}$, $T_\phi T_F f=P\phi PFf=P\phi Ff=T_{\phi F}f$. Similarly $T_{F^*}T_\phi=T_{F^*\phi}$.

THEOREM 3.9. *Let $\phi\in H^\infty_{n\times n}$, then T_ϕ is invertible if and only if $\phi^{-1}\in H^\infty_{n\times n}$ in which case $T_\phi^{-1}=T_{\phi^{-1}}$.*

PROOF. If T_ϕ is invertible, then $\phi^{-1}\in L^\infty_{n\times n}$ and $\phi=G^*H$ where G, G^{-1}, H, $H^{-1}\in H^2_{n\times n}$. Since $G^*=\phi H^{-1}\in H^2_{n\times n}$, G is constant and $\phi^{-1}=H^{-1}(G^{-1})^*\in H^\infty_{n\times n}$.

Conversely, if $\phi^{-1}\in H^\infty_{n\times n}$, then $T_\phi T_{\phi^{-1}}=T_{\phi^{-1}}T_\phi=T_I=I$.

COROLLARY 3.10. *Let $\phi\in H^\infty_{n\times n}$. If any one of the operators T_ϕ, T_{ϕ^*}, $T_{\phi'}$, $T_{\bar\phi}$ or $T_{\det \phi}$ is invertible, then the other four are also invertible.*

If $\phi\in L^\infty_{n\times n}$ is unitary, then $\phi^*=\phi^{-1}$. Thus T_ϕ invertible implies $\phi=G^*H$ where G, G^{-1}, H, $H^{-1}\in H^2_{n\times n}$ and $HH^*=(GG^*)^{-1}$.

The theorem which will follow demonstrates that it is sufficient to consider T_ϕ for the case where ϕ is unitary. This result can be exploited to yield a set of necessary conditions on ϕ for invertibility of T_ϕ as well as a different set of sufficiency conditions. These resemble to an extent the results given by Devinatz [1] for the scalar case. Since the preliminary development required is quite different from what is given here, these results will not be included in this paper.

THEOREM 3.11. *If $\phi\in L^\infty_{n\times n}$ and T_ϕ is invertible, then there exists a factorization $\phi=UK$ where U is unitary, $K\in H^\infty_{n\times n}$ is outer, and both T_K and T_U are invertible.*

PROOF. Now $\phi^*\phi\in L^\infty_{n\times n}$ is positive definite and 3.5 implies $\int\log|\det\phi|\,d\mu>-\infty$. Thus $\phi^*\phi=K^*K$ where $K\in H^\infty_{n\times n}$ is outer, see [5, p. 193]. Let $U=(\phi^{-1})^*K^*$, then U is unitary. Since $\phi^{-1}\in L^\infty_{n\times n}$, $K^{-1}\in L^\infty_{n\times n}$, so $K^{-1}\in H^\infty_{n\times n}$. Also $\phi=UK$. Finally, $T_\phi=T_UT_K$. Since T_ϕ and T_K are invertible, T_U is invertible.

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