

## SOME CHARACTERIZATIONS OF THE FREUDENTHAL COMPACTIFICATION OF A SEMICOMPACT SPACE

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Throughout this paper let  $X$  denote a semicompact Hausdorff space (i.e. every point of  $X$  has arbitrarily small neighborhoods with compact boundaries). For completeness we include the definition of the Freudenthal compactification of  $X$  [3] as formulated by K. Morita in [4]. Let  $\mathfrak{M}$  be the set of all finite open coverings  $\{G_1, G_2, \dots, G_n\}$  of  $X$  such that  $F_r G_i$  are compact. Then  $X$  is a completely regular space and  $\mathfrak{M}$  is a completely regular uniformity of  $X$  agreeing with the topology of  $X$ . Let  $S$  be the completion of  $X$  with respect to  $\mathfrak{M}$ . Then  $S$  is a compact Hausdorff space containing  $X$  as a dense subset having the following properties:

(a) For any point  $x$  of  $S$  and any open set  $R$  of  $S$  containing  $x$  there is an open set  $V$  of  $S$  containing  $x$  such that  $V \subset R$  and  $F_r V \subset X$ .

(b) Any two disjoint closed subsets of  $X$  with compact boundaries have disjoint closures in  $S$ .

K. Morita has shown that  $S$  is the topologically unique Hausdorff compactification of  $X$  satisfying (a) and (b), (i.e. if  $C$  is any compact Hausdorff space containing  $X$  as a dense subset and if  $C$  satisfies (a) and (b) then there is a homeomorphism of  $S$  onto  $C$  that leaves points of  $X$  fixed) [4]. This topologically unique space  $S$  is denoted by  $\gamma X$  and is called the Freudenthal compactification of  $X$  after H. Freudenthal who first defined it [3]. For further properties see [3], [4] or [6].

The purpose of this paper is to characterize  $\gamma X$  by means of certain minimum and maximum properties similar to those which characterize the Stone-Ćech compactification of  $\beta X$  of  $X$ .

First we need some definitions. Let  $\mathfrak{a}$  denote the set of all mappings of  $X$  into  $I = [0, 1]$ . For each  $f \in \mathfrak{a}$ , let  $B(f) = \{t \in I: F_r f^{-1}(t) \text{ contains a compact set } K \text{ that separates } X \text{ into two disjoint open sets } M \text{ and } N \text{ where } f(M) \subset [0, t] \text{ and } f(N) \subset [t, 1]\}$ . Note that in particular  $B(f)$  contains all of those points in  $I$  whose point inverse has an empty or compact boundary. Finally let  $F = \{f \in \mathfrak{a}: B(f) \text{ is dense in } I\}$ .

LEMMA 1. *Every  $f \in F$  has a unique continuous extension  $f$  to  $\gamma X$ .*

PROOF. Let  $f \in F$  and let  $\{H_1, H_2, \dots, H_k\}$  be a finite open cover-

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Presented to the Society, January 24, 1967; received by the editors February 23, 1967.

ing of  $I$ . Since  $B(f)$  is dense in  $I$  there exists a finite open cover  $\{K_1, K_2, \dots, K_k\}$  of  $I$  such that  $\overline{K}_i \subset H_i$  and  $FrK_i$  is a finite subset of  $B(f)$ . It then follows from our definition of  $B(f)$  that for each  $i$  there exists an open set  $G_i$  of  $X$  such that  $f^{-1}(K_i) \subset G_i \subset f^{-1}(H_i)$  and  $FrG_i$  is compact. Thus if we consider  $X$  as a uniform space with uniformity  $\mathfrak{M}$ , we have shown that  $f$  is uniformly continuous. Hence  $f$  is uniformly continuous and  $f$  can be extended to a unique continuous mapping  $f'$  of  $\gamma X$  into  $I$ .

The above proof is a modification of the proof of theorem 3 in [5] where K. Morita proves the following: Let  $f$  be a closed mapping of a semicompact metric space  $X$  onto a semicompact metric space  $Y$ . Then  $f$  can be extended to a continuous mapping of  $\gamma X$  onto  $\gamma Y$ .

REMARK. In [1, Theorem (3.1)] the author proved the following: Let  $X$  be a locally connected generalized continuum. Then  $X$  has the property that the complement of every compact set in  $X$  has at most one nonconditionally compact component if and only if every  $f \in F$  has a continuous extension to  $X_\infty$ , the one-point compactification of  $X$ . One can show that  $\gamma X$  is topologically equivalent to  $X_\infty$  if and only if  $X$  has the property that the complement of every compact in  $X$  has at most one nonconditionally compact component. Thus  $\gamma X$  is topologically equivalent to  $X_\infty$  if every  $f \in F$  has a continuous extension to  $X_\infty$ .

LEMMA 2. *For every pair of disjoint closed subsets  $A$  and  $B$  of  $\gamma X$  there exist an  $f \in F$  and a continuous extension  $f'$  of  $f$  to  $\gamma X$  such that  $f'(A) = 0$  and  $f'(B) = 1$ .*

PROOF. We will proceed to construct a mapping  $f'$  on  $\gamma X$  separating  $A$  and  $B$  such that  $f'|X = f$  is in  $F$ . Let  $D$  denote the set of positive dyadic rational numbers. Let  $R_1 = X - B$  and  $R_0$  denote an open subset of  $\gamma X$  containing  $A$  such that the closure of  $R_0$  misses  $B$  and the boundary of  $R_0$  lies entirely in  $X$ . We then proceed as in the proof of Urysohn's Lemma in [7, p. 115] and select for each  $t \in D$  an open set  $R_t$  of  $\gamma X$  containing  $A$  such that  $R_t = X$  for  $t > 1$ , and  $\overline{R}_s \subset R_t$  whenever  $s, t$  are in  $D$  and  $s < t$  but with the additional requirement that for each  $t \in D$ ,  $FrR_t \subset X$ . Then if we define  $f'(x) = \inf \{t: x \in R_t\}$  we have that  $f'$  is a continuous function such that  $f'(A) = 0$ ,  $f'(B) = 1$  and for each  $t \in D$ ,  $FrR_t \subset Frf'^{-1}(t)$ . Hence  $f = f'|X$  is in  $F$  as required.

THEOREM 1. *The Freudenthal compactification  $\gamma X$  of  $X$  satisfies the following:*

- (i) *Every  $f \in F$  has a unique continuous extension  $f'$  to  $\gamma X$ ; and*
- (ii) *For every pair of distinct points  $x$  and  $y$  in the closure of  $\gamma X - X$  there exists an  $f \in F$  such that  $f'(x) \neq f'(y)$ .*

Furthermore  $\gamma X$  is the topologically unique smallest Hausdorff compactification of  $X$  satisfying (i) and  $\gamma X$  is the topologically unique largest Hausdorff compactification of  $X$  satisfying (ii). Hence  $\gamma X$  is the topologically unique Hausdorff compactification of  $X$  satisfying (i) and (ii).

PROOF. By Lemmas 1 and 2,  $\gamma X$  satisfies (i) and (ii). The minimality, maximality and uniqueness of  $\gamma X$  follows immediately from the results in §5 in [2].

REMARK. Let  $P$  denote the product space  $I^F$  and define an embedding  $e$  of  $X$  into  $P$  by  $(\Pi_f \circ e)(x) = f(x)$  for  $x \in X$  and  $f \in F$ . (Here  $\Pi_f$  is the projection of  $P$  onto the  $f$ th coordinate.) Then if  $\psi X$  denotes the closure of  $e(X)$  in  $P$ ,  $\psi X$  is a Hausdorff compactification of  $X$  satisfying (i) and (ii). Thus  $\psi X$  and  $\gamma X$  are topologically equivalent and we have shown that the Freudenthal compactification of a semi-compact Hausdorff space can be obtained as a Tychonoff-type embedding.

When  $X$  is a locally compact Hausdorff space one can replace  $F$  by the set of all (closed) mappings of  $X$  into  $I$  that have compact boundaries of point inverses.

The author has shown that when  $X$  is locally compact there exists a Hausdorff compactification  $\alpha_F X$  of  $X$  satisfying (i) and (ii) with respect to any arbitrary family of mappings of  $X$  into any compact Hausdorff space  $Y$ . Thus it may be of interest to investigate the properties of compactifications one obtains when  $F$  is replaced by certain other classes of mappings.

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