SOME CHARACTERIZATIONS OF THE FREUDENTHAL COMPACTIFICATION OF A SEMICOMPACT SPACE

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Throughout this paper let $X$ denote a semicompact Hausdorff space (i.e. every point of $X$ has arbitrarily small neighborhoods with compact boundaries). For completeness we include the definition of the Freudenthal compactification of $X$ [3] as formulated by K. Morita in [4]. Let $\mathcal{M}$ be the set of all finite open coverings \{${G_1, G_2, \cdots, G_n}$\} of $X$ such that $FrG_i$ are compact. Then $X$ is a completely regular space and $\mathcal{M}$ is a completely regular uniformity of $X$ agreeing with the topology of $X$. Let $S$ be the completion of $X$ with respect to $\mathcal{M}$. Then $S$ is a compact Hausdorff space containing $X$ as a dense subset having the following properties:

(a) For any point $x$ of $S$ and any open set $R$ of $S$ containing $x$ there is an open set $V$ of $S$ containing $x$ such that $V \subset R$ and $FrV \subset X$.

(b) Any two disjoint closed subsets of $X$ with compact boundaries have disjoint closures in $S$.

K. Morita has shown that $S$ is the topologically unique Hausdorff compactification of $X$ satisfying (a) and (b), (i.e. if $C$ is any compact Hausdorff space containing $X$ as a dense subset and if $C$ satisfies (a) and (b) then there is a homeomorphism of $S$ onto $C$ that leaves points of $X$ fixed) [4]. This topologically unique space $S$ is denoted by $\gamma X$ and is called the Freudenthal compactification of $X$ after H. Freudenthal who first defined it [3]. For further properties see [3], [4] or [6].

The purpose of this paper is to characterize $\gamma X$ by means of certain minimum and maximum properties similar to those which characterize the Stone-Cech compactification of $\beta X$ of $X$.

First we need some definitions. Let $\mathcal{C}$ denote the set of all mappings of $X$ into $I = [0, 1]$. For each $f \in \mathcal{C}$, let $B(f) = \{t \in I: Frf^{-1}(t)$ contains a compact set $K$ that separates $X$ into two disjoint open sets $M$ and $N$ where $f(M) \subset [0, t]$ and $f(N) \subset [t, 1]\}$. Note that in particular $B(f)$ contains all of those points in $I$ whose point inverse has an empty or compact boundary. Finally let $F = \{f \in \mathcal{C}: B(f)$ is dense in $I\}$.

**Lemma 1.** Every $f \in F$ has a unique continuous extension $f$ to $\gamma X$.

**Proof.** Let $f \in F$ and let $\{H_1, H_2, \cdots, H_k\}$ be a finite open cover-
ing of \( I \). Since \( B(f) \) is dense in \( I \) there exists a finite open cover \( \{K_1, K_2, \ldots, K_k\} \) of \( I \) such that \( \overline{K_i} \subset H_i \) and \( \text{Fr}K_i \) is a finite subset of \( B(f) \). It then follows from our definition of \( B(f) \) that for each \( i \) there exists an open set \( G_i \) of \( X \) such that \( f^{-1}(K_i) \subset G_i \subset f^{-1}(H_i) \) and \( \text{Fr}G_i \) is compact. Thus if we consider \( X \) as a uniform space with uniformity \( \mathcal{U} \), we have shown that \( f \) is uniformly continuous. Hence \( f \) is uniformly continuous and \( f \) can be extended to a unique continuous mapping \( f' \) of \( \gamma X \) into \( I \).

The above proof is a modification of the proof of theorem 3 in [5] where K. Morita proves the following: Let \( f \) be a closed mapping of a semicompact metric space \( X \) onto a semicompact metric space \( Y \). Then \( f \) can be extended to a continuous mapping of \( \gamma X \) onto \( \gamma Y \).

**Remark.** In [1, Theorem (3.1)] the author proved the following: Let \( X \) be a locally connected generalized continuum. Then \( X \) has the property that the complement of every compact set in \( X \) has at most one nonconditionally compact component if and only if every \( f \in F \) has a continuous extension to \( X_\infty \), the one-point compactification of \( X \). One can show that \( \gamma X \) is topologically equivalent to \( X_\infty \) if and only if \( X \) has the property that the complement of every compact in \( X \) has at most one nonconditionally compact component. Thus \( \gamma X \) is topologically equivalent to \( X_\infty \) if every \( f \in F \) has a continuous extension to \( X_\infty \).

**Lemma 2.** For every pair of disjoint closed subsets \( A \) and \( B \) of \( \gamma X \) there exist an \( f \in F \) and a continuous extension \( f' \) of \( f \) to \( \gamma X \) such that \( f'(A) = 0 \) and \( f'(B) = 1 \).

**Proof.** We will proceed to construct a mapping \( f' \) on \( \gamma X \) separating \( A \) and \( B \) such that \( f'|X = f \) is in \( F \). Let \( D \) denote the set of positive dyadic rational numbers. Let \( R_1 = X - B \) and \( R_0 \) denote an open subset of \( \gamma X \) containing \( A \) such that the closure of \( R_0 \) misses \( B \) and the boundary of \( R_0 \) lies entirely in \( X \). We then proceed as in the proof of Urysohn’s Lemma in [7, p. 115] and select for each \( t \in D \) an open set \( R_t \) of \( \gamma X \) containing \( A \) such that \( R_t = X \) for \( t > 1 \), and \( \overline{R_s} \subset R_t \) whenever \( s, t \) are in \( D \) and \( s < t \) but with the additional requirement that for each \( t \in D \), \( \text{Fr}R_t \subset X \). Then if we define \( f'(x) = \inf \{t : x \in R_t\} \) we have that \( f' \) is a continuous function such that \( f'(A) = 0 \), \( f'(B) = 1 \) and for each \( t \in D \), \( \text{Fr}R_t \subset \text{Fr}f'^{-1}(t) \). Hence \( f = f'|X \) is in \( F \) as required.

**Theorem 1.** The Freudenthal compactification \( \gamma X \) of \( X \) satisfies the following:

(i) Every \( f \in F \) has a unique continuous extension \( f' \) to \( \gamma X \); and

(ii) For every pair of distinct points \( x \) and \( y \) in the closure of \( \gamma X - X \) there exists an \( f \in F \) such that \( f'(x) \neq f'(y) \).
Furthermore \( \gamma X \) is the topologically unique smallest Hausdorff compactification of \( X \) satisfying (i) and \( \gamma X \) is the topologically unique largest Hausdorff compactification of \( X \) satisfying (ii). Hence \( \gamma X \) is the topologically unique Hausdorff compactification of \( X \) satisfying (i) and (ii).

**Proof.** By Lemmas 1 and 2, \( \gamma X \) satisfies (i) and (ii). The minimality, maximality and uniqueness of \( \gamma X \) follows immediately from the results in §5 in [2].

**Remark.** Let \( P \) denote the product space \( I^F \) and define an embedding \( e \) of \( X \) into \( P \) by \( (\Pi_i \circ e)(x) = f(x) \) for \( x \in X \) and \( f \in F \). (Here \( \Pi_i \) is the projection of \( P \) onto the \( i \)th coordinate.) Then if \( \psi X \) denotes the closure of \( e(X) \) in \( P \), \( \psi X \) is a Hausdorff compactification of \( X \) satisfying (i) and (ii). Thus \( \psi X \) and \( \gamma X \) are topologically equivalent and we have shown that the Freudenthal compactification of a semicompact Hausdorff space can be obtained as a Tychonoff-type embedding.

When \( X \) is a locally compact Hausdorff space one can replace \( F \) by the set of all (closed) mappings of \( X \) into \( I \) that have compact boundaries of point inverses.

The author has shown that when \( X \) is locally compact there exists a Hausdorff compactification \( \alpha_F X \) of \( X \) satisfying (i) and (ii) with respect to any arbitrary family of mappings of \( X \) into any compact Hausdorff space \( Y \). Thus it may be of interest to investigate the properties of compactifications one obtains when \( F \) is replaced by certain other classes of mappings.

**References**