SOME CHARACTERIZATIONS OF THE FREUDENTHAL COMPACTIFICATION OF A SEMICOMPACT SPACE

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Throughout this paper let X denote a semicompact Hausdorff space (i.e. every point of X has arbitrarily small neighborhoods with compact boundaries). For completeness we include the definition of the Freudenthal compactification of X [3] as formulated by K. Morita in [4]. Let $\mathfrak{M}$ be the set of all finite open coverings $\{G_1, G_2, \cdots, G_n\}$ of X such that F_rG_i are compact. Then X is a completely regular space and $\mathfrak{M}$ is a completely regular uniformity of X agreeing with the topology of X. Let S be the completion of X with respect to $\mathfrak{M}$. Then S is a compact Hausdorff space containing X as a dense subset having the following properties:

(a) For any point x of S and any open set R of S containing x there is an open set V of S containing x such that $V \subset R$ and $Fr\, V \subset X$.

(b) Any two disjoint closed subsets of X with compact boundaries have disjoint closures in S.

K. Morita has shown that S is the topologically unique Hausdorff compactification of X satisfying (a) and (b), (i.e. if C is any compact Hausdorff space containing X as a dense subset and if C satisfies (a) and (b) then there is a homeomorphism of S onto C that leaves points of X fixed) [4]. This topologically unique space S is denoted by γX and is called the Freudenthal compactification of X after H. Freudenthal who first defined it [3]. For further properties see [3], [4] or [6].

The purpose of this paper is to characterize γX by means of certain minimum and maximum properties similar to those which characterize the Stone-Čech compactification of βX of X.

First we need some definitions. Let $\mathfrak{A}$ denote the set of all mappings of X into $I = [0, 1]$. For each $f \in \mathfrak{A}$, let $B(f) = \{t \in I$: $Frf^{-1}(t)$ contains a compact set K that separates X into two disjoint open sets M and N where $f(M) \subset [0, t]$ and $f(N) \subset [t, 1]\}$. Note that in particular $B(f)$ contains all of those points in I whose point inverse has an empty or compact boundary. Finally let $F = \{f \in \mathfrak{A}$: $B(f)$ is dense in $I\}$.

Lemma 1. *Every $f \in F$ has a unique continuous extension f to γX.*

Proof. Let $f \in F$ and let $\{H_1, H_2, \cdots, H_k\}$ be a finite open cover-

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ing of I. Since $B(f)$ is dense in I there exists a finite open cover $\{K_1, K_2, \cdots, K_k\}$ of I such that $\overline{K}_i \subset H_i$ and FrK_i is a finite subset of $B(f)$. It then follows from our definition of $B(f)$ that for each i there exists an open set G_i of X such that $f^{-1}(K_i) \subset G_i \subset f^{-1}(H_i)$ and FrG_i is compact. Thus if we consider X as a uniform space with uniformity $\mathfrak{M}$, we have shown that f is uniformly continuous. Hence f is uniformly continuous and f can be extended to a unique continuous mapping f' of γX into I.

The above proof is a modification of the proof of theorem 3 in [5] where K. Morita proves the following: Let f be a closed mapping of a semicompact metric space X onto a semicompact metric space Y. Then f can be extended to a continuous mapping of γX onto γY.

Remark. In [1, Theorem (3.1)] the author proved the following: Let X be a locally connected generalized continuum. Then X has the property that the complement of every compact set in X has at most one nonconditionally compact component if and only if every $f \in F$ has a continuous extension to X_∞, the one-point compactification of X. One can show that γX is topologically equivalent to X_∞ if and only if X has the property that the complement of every compact in X has at most one nonconditionally compact component. Thus γX is topologically equivalent to X_∞ if every $f \in F$ has a continuous extension to X_∞.

Lemma 2. *For every pair of disjoint closed subsets A and B of γX there exist an $f \in F$ and a continuous extension f' of f to γX such that $f'(A)=0$ and $f'(B)=1$.*

Proof. We will proceed to construct a mapping f' on γX separating A and B such that $f'|X=f$ is in F. Let D denote the set of positive dyadic rational numbers. Let $R_1 = X - B$ and R_0 denote an open subset of γX containing A such that the closure of R_0 misses B and the boundary of R_0 lies entirely in X. We then proceed as in the proof of Urysohn's Lemma in [7, p. 115] and select for each $t \in D$ an open set R_t of γX containing A such that $R_t = X$ for $t > 1$, and $\overline{R}_s \subset R_t$ whenever s, t are in D and $s < t$ but with the additional requirement that for each $t \in D$, $FrR_t \subset X$. Then if we define $f'(x) = \inf\{t : x \in R_t\}$ we have that f' is a continuous function such that $f'(A)=0$, $f'(B)=1$ and for each $t \in D$, $FrR_t \subset Frf'^{-1}(t)$. Hence $f = f'|X$ is in F as required.

Theorem 1. *The Freudenthal compactification γX of X satisfies the following*:

(i) *Every $f \in F$ has a unique continuous extension f' to γX*; *and*

(ii) *For every pair of distinct points x and y in the closure of $\gamma X - X$ there exists an $f \in F$ such that $f'(x) \neq f'(y)$.*

Furthermore γX is the topologically unique smallest Hausdorff compactification of X satisfying (i) *and γX is the topologically unique largest Hausdorff compactification of X satisfying* (ii). *Hence γX is the topologically unique Hausdorff compactification of X satisfying* (i) *and* (ii).

PROOF. By Lemmas 1 and 2, γX satisfies (i) and (ii). The minimality, maximality and uniqueness of γX follows immediately from the results in §5 in [2].

REMARK. Let P denote the product space I^F and define an embedding e of X into P by $(\Pi_f \circ e)(x) = f(x)$ for $x \in X$ and $f \in F$. (Here Π_f is the projection of P onto the fth coordinate.) Then if ψX denotes the closure of $e(X)$ in P, ψX is a Hausdorff compactification of X satisfying (i) and (ii). Thus ψX and γX are topologically equivalent and we have shown that the Freudenthal compactification of a semicompact Hausdorff space can be obtained as a Tychonoff-type embedding.

When X is a locally compact Hausdorff space one can replace F by the set of all (closed) mappings of X into I that have compact boundaries of point inverses.

The author has shown that when X is locally compact there exists a Hausdorff compactification $\alpha_F X$ of X satisfying (i) and (ii) with respect to any arbitrary family of mappings of X into any compact Hausdorff space Y. Thus it may be of interest to investigate the properties of compactifications one obtains when F is replaced by certain other classes of mappings.

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