A NOTE ON AN EXAMPLE OF STALLINGS[1]

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1. **Introduction.** R. H. Bing [1] has shown that Euclidean 3-space E^3 does not contain an uncountable collection of pairwise disjoint wild closed surfaces. J. Stallings [3] gave an example of an uncountable collection of pairwise disjoint wild disks in E^3. J. Martin [2] showed that in an uncountable collection of pairwise disjoint disks all but a countable number of elements of the collection lie on a 2-sphere. In a review of Martin's paper, O. G. Harrold pointed out that what is apparently unknown is whether an uncountable collection of pairwise disjoint disks must contain a pair of equivalently imbedded elements. The purpose of this note is to show that it need not.

We use E^3 to denote the set of all 3-tuples (x_1, x_2, x_3) of real numbers with the usual topology. We identify E^1 (the reals) with the x_1-axis in E^3, and E^2 (the plane) with the x_1x_2-coordinate plane. We use coordinates (x_1, x_2) rather than $(x_1, x_2, 0)$ to represent points of E^2.

Compact sets H and K are *equivalently imbedded* in E^3 if there is a homeomorphism of E^3 onto itself which carries H onto K. If there is a polyhedron P so that H and P are equivalently imbedded, we say that H is *tame*. Otherwise, H is *wild*. The set H is *locally tame* at $p\in H$ if there is a neighborhood N of p so that the closure of $N\cap H$ is tame. If W is a closed subset of H and H is locally tame at each point of $H-W$, H is said to be *locally tame modulo* W.

If D is a disk or 3-cell, Bd D will denote the boundary simple closed curve or 2-sphere of D. We use Cl H for the closure of the point set H.

2. **Construction.** Let M be the usual "middle-third" Cantor set in $[0, 1]\subset E^1$. That is, $M=\bigcup_{i=0}^{\infty} K_i$ where $K_0=[0, 1]$ and K_i is the union of the 2^i pairwise disjoint closed intervals obtained by removing the middle-third open interval from each component of K_{i-1} if $i=1, 2, 3, \cdots$. If $i=1, 2, 3, \cdots$, we shall denote the 2^i components of K_i by $L_1^i, L_2^i, L_3^i, \cdots, L_{2^i}^i$ where each point of L_j^i precedes each point of L_{j+1}^i if $j=1, 2, \cdots$ or 2^i-1. We denote $L_j^i\cap M$ by M_j^i.

We now define a sequence $N_1, N_2, N_3, \cdots$ of Cantor sets in E^2 so that

$$N = \mathrm{Cl}\left(\bigcup_{i=1}^{\infty} N_i\right) = M\cup\left(\bigcup_{i=1}^{\infty} N_i\right)$$

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is a Cantor set. The set N_i is defined to be

$$\left[\bigcup_{k=1}^{2^{i-1}} (M_{2k-1}^{i} \times \{2/3^{i}\})\right] \cup \left[\bigcup_{k=1}^{2^{i-1}} (M_{2k}^{i} \times M_{2}^{i})\right].$$

Now let $\{S_\alpha\}$ be the set of all infinite sequences each of whose terms is either 0 or 1. If S_α is the sequence $m_1, m_2, m_3, \cdots$, we associate the sequence $L_{\bar{m}_1}^1 \supset L_{\bar{m}_2}^2 \supset L_{\bar{m}_3}^3 \supset \cdots$ with S_α where $\bar{m}_j$ is odd if m_j is 0 and even if m_j is 1. The correspondence $S_\alpha \to \bigcap_{i=1}^\infty L_{\bar{m}_i}^i = p_\alpha \in M$ is a 1-1 correspondence between the elements of $\{S_\alpha\}$ and the points of M. Notice that the subset $\{S'_\alpha\}$ of $\{S_\alpha\}$, consisting of those members of $\{S_\alpha\}$ in which no 1's appear adjacent to one another, has cardinality of the continuum.[2]

Now let Q be a 3-cell in E^3 such that Bd Q is locally tame modulo a Cantor set C and so that any arc on Bd Q which intersects C is wild. An example of such a cell is described by Stallings in [3]. There is a homeomorphism $h\colon Q \to Q' = \{(x_1, x_2, x_3) \mid 0 \leqq x_i \leqq 1,\ i = 1, 2, 3\}$ so that $h(C) = N$. Let D'_α be the intersection of Q' with the plane $x_1 = p'_\alpha$ where p'_α is associated with the sequence S'_α, and let $D_\alpha = h^{-1}(D'_\alpha)$.

The set $\{D_\alpha\}$ is an uncountable collection of pairwise disjoint wild disks in E^3. The boundary of each D_α is locally tame modulo $D_\alpha \cap C$. A description of the wild set of Bd D_α is given by examining the sequence S'_α. To each 0 in S'_α there corresponds an isolated wild point of Bd D_α, and to each 1 in S'_α there corresponds a Cantor set of wild points, and S'_α describes the relative order of these "components" of the wild set on Bd D_α. The fact that a space homeomorphism taking one disk onto another must necessarily carry wild points to wild points and the choice of the sequences $\{S'_\alpha\}$ shows that no pair of the disks $\{D_\alpha\}$ are equivalently imbedded in E^3.

References

1. R. H. Bing, *E^3 does not contain uncountably many mutually exclusive wild surfaces*, Abstract 801t, Bull. Amer. Math. Soc. **63** (1957), 404.

2. J. Martin, *A note on uncountably many disks*, Pacific J. Math. **13** (1963), 1331–1333.

3. J. R. Stallings, *Uncountably many wild disks*, Ann. of Math. (2) **71** (1960), 185–186.

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[2] In fact, as pointed out by the referee, the image of the set $\{S'_\alpha\}$ under the 1-1 correspondence $S_\alpha \to p_\alpha$ is a Cantor set thus showing that there is an uncountable compact collection of pairwise disjoint disks, no two of which are equivalently imbedded.