ON THE METASTABLE HOMOTOPIE OF $O(n)$

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The purpose of this note is to present an alternate proof of the main result of [3]. In particular, we prove

**Theorem 1 (1 of [3])**. If $k > 4$, a nontrivial stable real vector bundle over $S^k$ is the sum of an irreducible $(2k + 1)$-plane bundle and a trivial bundle.

This result has several geometric applications and easily implies

**Theorem 2 (2 of [3])**. For $q < 2(n - 1)$ and $n \geq 13$,

$$\pi_q(O(n)) = \pi_q(O) \oplus \pi_{q-1}(V_{2n,n}).$$

We will be only concerned with the proof of Theorem 1 in this note. Extensive calculation of $\pi_{q-1}(V_{2n,n})$ are given in [4].

The proof will be preceded by several lemmas which use the following notation. Let $X$ be a space. The symbol $X[k]$ means the $(k - 1)$-connected fibering over $X$. $E^r,t_i(X)$ is the $r$th term in the Adams spectral sequence [1] for $X$ leading to the group associated with $\pi_i^s(X)$. Thus $E^r,t_i(X) = \text{Ext}_{\pi_i^s}^r(\tilde{H}^*(X; Z_2), Z_2)$, where $A$ is the mod 2 Steenrod algebra. $V_k$ is the fiber of $BO_k \to BO$.

**Lemma 1**. If $t - s \leq 4k$ and $k > 4$, then

$$E^2_{s,t}(BSO_{2k+1}[2k + 1]) \simeq E^2_{s,t}(BSO[2k + 1]) \oplus E^2_{s,t}(V_{2k+1}).$$

**Proof**. Let $p$ be the smallest integer such that $2^p > 4k$. For $p = 4a + b$, $0 \leq b \leq 3$, let $j(p) = 8a + 2^b$. Let $i_p : BSO[j(p)] \to BSO$ be the usual inclusion. Then $i^*_p w_j = 0$ for all $j < 2^p$ [5]. If $k > 4$, then $BSO_{2k+1}[j(p)]$ is the total space of $i_p^* \gamma_{2k+1}$ where $\gamma_{2k+1}$ is the bundle $BSO_{2k+1} \to BSO$. Therefore

$$H^q(BSO_{2k+1}[j(p)]) \simeq \sum_{u+v=q} H^u(BSO[j(p)]) \oplus H^v(V_{2k+1})$$

for $q \leq 4k + 1$ as $Z_2$ modules. Since $i' : BSO[2k+1] \to BSO[j(p)]$ induces the zero map in cohomology if $2k + 1 > j(p)$ [5], we have

$$H^q(BSO_{2k+1}[2k + 1]) \simeq \sum_{u+v=q} H^u(BSO[2k + 1]) = H^v(V_{2k+1})$$

as $A$ modules. This implies the lemma.

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Corollary 2. The projection \( \bar{p} : BSO_{2k+1}[2k+1] \to BSO[2k+1] \) induces an epimorphism

\[ \bar{p}^*: E_2^{s,t}(BSO_{2k+1}[2k+1]) \to E_2^{s,t}(BSO[2k+1]). \]

Lemma 3. Let \( \omega_k : S^k \to BSO[2k+1] \) be a generator of \( \pi_{4k}(BSO[2k+1]) \).

\( \omega_k \) has filtration

\[
\begin{align*}
&\geq k-1, & k &= 0, 1 \pmod{4}, \\
&\geq k-2, & k &= 2 \pmod{4}, \text{ and} \\
&\geq k, & k &= 3 \pmod{4}
\end{align*}
\]

in the Adams spectral sequence sense.

Proof. Consider the diagram

\[ S^k \xrightarrow{i_j} BSO[4k] \to \cdots \xrightarrow{i_1} BSO[2k \times 1]. \]

In cohomology each map \( i^*_u \) is zero if the connectivity increases. Maps between spaces which induce the zero map in cohomology have filtration \( \geq 1 \). Thus the question of filtration reduces to only counting the number of nonzero homotopy groups between \( 2k+1 \) and \( 4k \) in \( BSO \).

Lemma 4. \( E_2^{s,t}(V_{2k+1}) = 0 \) if \( t-s = 4k-1 \) and

\[
\begin{align*}
s &\geq k, & k &= 0, 1 \pmod{2} \\
&\geq k + 1, & k &= 2, 3 \pmod{4}
\end{align*}
\]

Proof. Let \( 0 \leq u < 8 \) be such that \( 2k+1+u = 4k-1 \) (mod 8). Let \( 0 \leq v < 4 \) be such that \( k = v \) (mod 4). It is an easy calculation to compute \( \text{Ext}_A^{s,t}(\tilde{V}(V_{2k+1}), Z_2) \) by minimal resolution [1] for \( t = 2k+1 + u + s \) and \( s = v, v+1 \) (or one can use the tables in [4]). The Adams periodicity theorems [2] enable one to use this calculation to prove the lemma.

Now we can prove the main theorem if \( k \neq 2 \) (mod 4).

Proof of Theorem 1. Let \( \bar{\alpha} \in E_2^{s,t}(BSO_{2k+1}[2k+1]) \) project to the class in \( E_2^{s,t}(BSO[2k+1]) \) to which \( [\omega_k] \) projects. We are finished if we show that \( \bar{\alpha} \) is a permanent cycle. Since \( \bar{p} \bar{\alpha} \) is not a boundary, \( \bar{\alpha} \) is not a boundary. If \( \delta_r \bar{\alpha} = \beta \), then \( \beta \) must be in the summand \( E_2^{s,t}(V_{2k+1}) \) but Lemma 4 says that this group is zero for all possible \( s \) and \( t \) values that \( \beta \) might have.

To finish the proof of Theorem 1, we just need to indicate how to fix the argument for \( k \equiv 2 \) (4). Let \( \alpha \in H^u(s)(V_{2k+1}) \) be a generating set over \( A \). Let

\[ BSO_{2k+1}[j(p)] \xrightarrow{f} BSO[j(p)] \times \prod_i K(Z_2, u(i)) \]
be the usual map on the first factor, and on the second factor \( f^*(\iota_i) = \alpha_i \) where \( \iota_i \) is the characteristic class of \( K_i = K(\mathbb{Z}_2, u(i)) \). Let \( F \) be the fiber of \( f \) and let \( \tilde{\pi} : E \to BSO[2k + 1] \) be the fiber space induced from \( f \) by the composite map

\[
BSO[2k + 1] \to BSO[j(\tilde{\pi})] \to BSO[j(\tilde{\pi})] \times \prod_i K_i,
\]

where the maps are the obvious ones. Following the arguments given above we can prove

**Lemma 5.** If \( t - s \leq 4k \) and \( k > 4 \), then:

(i) \( E_2^{s,t}(E) \simeq E_2^{s,t}(BSO[2k+1]) \oplus E_2^{s,t}(F) \);

(ii) \( E_2^{s,t}(F) \simeq E_2^{s+1,t+1}(V_{2k+1}) \).

The theorem follows from this as above.

**Bibliography**


5. R. E. Stong, *Determination of \( H^*(BO(k, \cdots, \infty), \mathbb{Z}_2) \) and \( H^*(BU(k, \cdots, \infty), \mathbb{Z}_2) \)*, Trans. Amer. Math. Soc. 107 (1963), 526–544.

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