THE DISTRIBUTION OF kTH POWER RESIDUES AND NONRESIDUES

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1. Introduction. In 1962 D. A. Burgess [**1**] established this general theorem concerning character sums:

THEOREM A. *If p is a prime and if χ is a nonprincipal Dirichlet character, modulo p, and if H and r are arbitrary positive integers then*

$$\sum_{m=n+1}^{n+H} \chi(m) \ll H^{1-1/(r+1)} p^{1/4r} \ln p \tag{I}$$

for any integer n, where $A \ll B$ is Vinogradov's notation for $|A| < cB$ for some constant c, and in this theorem c is absolute.

In this paper Theorem A will be used to improve a special case of the Vinogradov result [**3**]:

THEOREM B. *Let E_j be a class of kth power nonresidues for $j=1, 2, 3, \cdots, k$ and let E_0 be the class of kth power residues, $kt=p-1$. Also let $N_j(H)$ be the number of positive integers in E_j that are $\leqq H$. Then $N_j(H)=H/k+T_j$ where $T_j^2<T+p/2$ with*

$$T = \sum_{x=1}^{H} \sum_{(y,x)=1}^{p/x} (p/xy+1).$$

In particular, Theorem B implies that $T_j < \sqrt{p} \ln p$.

Specifically, in this paper the following is proved:

THEOREM. *Let E_j be the classes of kth power nonresidues, $j=1, 2, 3, \cdots, k-1$; and E_0 is the class of kth power residues, $kt=p-1$. Also let $N_j(H)$ be the number of positive integers in E_j that are $\leqq H$.*

Then $N_j(H)=H/k+T_j$ where $T_j \ll H^{1-1/(r+1)} p^{1/4r} \ln p$, r is a positive integer.

Notice that this theorem is significant for $p^{1/4+1/4r}(\ln p)^{r+1} < H$ and is an improvement of Theorem B for $H < p^{1/2+1/4r-1/4r^2}$. Theorem B has content only when $\sqrt{p} \ln p < H$.

In [2] the author proved this theorem for $k=3$ and 5 but erroneously referred to these as being special cases of Theorem B.

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2. **Proof of the Theorem.** Let χ be a kth power Dirichlet character and order the E_j so that $\chi(a)=\rho^j$ for all a in E_j, where ρ is a primitive kth root of unity. Let

$$N_j(H) = H/k + T_j. \tag{II}$$

Now since $\sum_{j=0}^{k-1} N_j(H)=\sum_{j=0}^{k-1}(H/k+T_j)=H+\sum_{j=0}^{k-1} T_j$ and trivially $\sum_{j=0}^{k-1} N_j(H)=H$, we have therefore

$$\sum_{j=0}^{k-1} T_j = 0. \tag{III}$$

Also

$$\sum_{m=1}^{H} \chi(m) = \sum_{j=0}^{k-1} \rho^j N_j(H) = \sum_{j=0}^{k-1} \rho^j H/k + \sum_{j=0}^{k-1} \rho^j T_j$$
$$= H/k \sum_{j=0}^{k-1} \rho^j + \sum_{j=0}^{k-1} \rho^j T_j = \sum_{j=0}^{k-1} \rho^j T_j.$$

And by Burgess' Theorem this implies

$$\left| \sum_{j=0}^{k-1} \rho^j T_j \right| < cH^{1-1/(r+1)} p^{1/4r} \ln p.$$

Now χ^t is also a nonprincipal Dirichlet character for $1 \leqq t \leqq k-1$, and since $\chi^t(a)=\rho^{tj}$ for a in E_j it follows that:

$$\sum_{m=1}^{H} \chi^t(m) = \sum_{j=0}^{k-1} \rho^{tj} N_j(H)$$
$$= \sum_{j=0}^{k-1} \rho^{tj} H/k + \sum_{j=0}^{k-1} \rho^{tj} T_j$$
$$= H/k \sum_{j=0}^{k-1} \rho^{tj} + \sum_{j=0}^{k-1} \rho^{tj} T_j = \sum_{j=0}^{k-1} \rho^{tj} T_j.$$

Applying Burgess' Theorem one has:

$$\left| \sum_{j=0}^{k-1} \rho^{tj} T_j \right| < c_j H^{1-1/(r+1)} p^{1/4r} \ln p, \qquad 1 \leqq t < k. \tag{IV}$$

Now for a specific E_j, say E_{j*}, consider expression (IV) divided by ρ^{j^*t} yielding:

$$\left| \sum_{j=0}^{k-1} \rho^{t(j-j^*)} T_j \right| < c_t H^{1-1/(r+1)} p^{1/4r} \ln p, \qquad 1 \leqq t < k. \tag{V}$$

Now summing over all expressions in (V) and throwing in expression (III) one has:

$$\text{(VI)} \qquad \left| \sum_{t=0}^{k-1} \sum_{j=0}^{k-1} \rho^{t(j-j^*)} T_j \right| \leqq \sum_{t=0}^{k-1} \left| \sum_{j=0}^{k-1} \rho^{t(j-j^*)} T_j \right| .$$

But

$$\begin{aligned} \sum_{t=0}^{k-1} \sum_{j=0}^{k-1} \rho^{t(j-j^*)} T_j &= \sum_{j=0}^{k-1} T_j \sum_{t=0}^{k-1} \rho^{t(j-j^*)} \\ &= \sum_{j=0;\, j\neq j^*}^{k-1} T_j \sum_{t=0}^{k-1} \rho^{t(j-j^*)} + \sum_{t=0}^{k-1} T_{j^*} \\ &= kT_{j^*}, \text{ since } \sum_{t=0}^{k-1} \rho^{t(j-j^*)} = 0 \quad \text{unless } j = j^*. \end{aligned}$$

Hence

$$\begin{aligned} |T_{j^*}| &< \frac{1}{k} \sum_{t=1}^{k-1} c_t H^{1-1/(r+1)} p^{1/4r} \ln p \\ &= c^* H^{1-1/(r+1)} p^{1/4r} \ln p. \end{aligned}$$

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